Extended Affine Lie Algebras and Extended Affine Weyl Groups

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In Partial Fulfillment of the Requirements

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Doctor of Philosophy

In the Department of Mathematics

of the University of Saskatchewan

By

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SUMMARY OF DISSERTATION

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of the requirements for the

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Extended Affine Lie Algebras and Extended Affine Weyl Groups

This thesis is about extended affine Lie algebras and extended affine Weyl groups. In Chapter I, we provide the basic knowledge necessary for the study of extended affine Lie algebras and related objects. In Chapter II, we show that the well-known twisting phenomena which appears in the realization of the twisted affine Lie algebras can be extended to the class of extended affine Lie algebras, in the sense that some extended affine Lie algebras (in particular nonsimply laced extended affine Lie algebras) can be realized as fixed point subalgebras of some other extended affine Lie algebras (in particular simply laced extended affine Lie algebras) relative to some finite order automorphism. We show that extended affine Lie algebras of type A_1, B, C and BC can be realized as twisted subalgebras of types $A_\ell (l \geq 2)$ and D algebras. Also we show that extended affine Lie algebras of type BC can be realized as twisted subalgebras of type C algebras. In Chapter III, the last chapter, we study the Weyl groups of reduced extended affine root systems. We start by describing the extended affine Weyl group as a semidirect product of a finite Weyl group and a Heisenberg-like normal subgroup. This provides a unique expression for the Weyl group elements which in turn leads to a presentation of the Weyl group, called a presentation by conjugation. Using a new notion, called the index, which is an invariant of the extended affine root systems, we show that one of the important features of finite and affine root systems (related to Weyl group) holds for the class of extended affine root systems. We also show that extended affine Weyl groups (of index zero) are homomorphic images of some indefinite Weyl groups where the homomorphism and its kernel are given explicitly.

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Abstract

This thesis is about extended affine Lie algebras and extended affine Weyl groups. In Chapter I, we provide the basic knowledge necessary for the study of extended affine Lie algebras and related objects. In Chapter II, we show that the well-known twisting phenomena which appears in the realization of the twisted affine Lie algebras can be extended to the class of extended affine Lie algebras, in the sense that some extended affine Lie algebras (in particular nonsimply laced extended affine Lie algebras) can be realized as fixed point subalgebras of some other extended affine Lie algebras (in particular simply laced extended affine Lie algebras) relative to some finite order automorphism. We show that extended affine Lie algebras of type A_1, B, C and BC can be realized as twisted subalgebras of types $A_\ell (l \geq 2)$ and D algebras. Also we show that extended affine Lie algebras of type BC can be realized as twisted subalgebras of type C algebras. In Chapter III, the last chapter, we study the Weyl groups of reduced extended affine root systems. We start by describing the extended affine Weyl group as a semidirect product of a finite Weyl group and a Heisenberg-like normal subgroup. This provides a unique expression for the Weyl group elements which in turn leads to a presentation of the Weyl group, called a presentation by conjugation. Using a new notion, called the index, which is an invariant of the extended affine root systems, we show that one of the important features of finite and affine root systems (related to Weyl group) holds for the class of extended affine root systems. We also show that extended affine Weyl groups (of index zero) are homomorphic images of some indefinite Weyl groups where the homomorphism and its kernel are given explicitly.

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Contents

Chapter 0

Introduction

In this thesis we study the class of extended affine Lie algebras (EALA's for short) and related objects. This class is an axiomatic generalization of the simple finite dimensional complex Lie algebras and the affine Kac-Moody algebras.

In 1968 V. Kac and R. Moody introduced independently a class of infinite dimensional Lie algebras, called Kac-Moody algebras. These infinite dimensional algebras have been studied intensively and shown to be powerful tools for the investigation of many apparently disconnected fields, both in mathematics and mathematical physics.

Extended affine Lie algebras, in their axiomatic forms, were introduced in [H-KT] (under the name quasi simple Lie algebras) in an attempt to generalize the affine Kac-Moody algebras which occupy the most remarkable place among the class of Kac-Moody algebras, because of their connection with different fields in mathematics and mathematical physics. EALA's and related objects have been studied in various forms since the mid eighties. In this regard and not claiming to be complete, we can name the works of [Sa1], [Sa2], [W], [H-KT], [Po], [BGK], [BGKN], [Kr], [AWW-L], [AABGP],[ABGP], [A], [Sa-T] and [Y].

Roughly speaking, an EALA is a complex Lie algebra satisfying the following axioms.

- It has an invariant symmetric nondegenerate bilinear form.
- It has a finite dimensional self-centralizing abelian subalgebra relative to which we obtain a root space decomposition.
- Elements of root spaces corresponding to nonisotropic roots act locally nilpotently via the adjoint representation on the whole algebra.

- The set of roots is a discrete set, is irreducible in the usual sense and nonisotropic roots are not isolated.

Finite dimensional simple Lie algebras and affine Kac-Moody algebras are examples of EALA's. It is shown in [AABGP, Chapter I] that the set of roots of an EALA satisfies some axioms (see Definition I.1.14) which are a natural generalization of the axioms for a finite irreducible root system given in [Bou] or [Hum]. An EARS is defined to be a subset of a real vector space satisfying these axioms. Beside the finite irreducible root systems the root systems of affine Kac-Moody algebras are also examples of EARS's. More generally, the toroidal Lie algebras (possibly with certain derivations added) studied in [BC], [EMY1,2] and [EM] also have root systems which are EARS's. We remark here that there are EARS's which are not the root system of any EALA. For each EARS, there is a finite root system attached to it of unique type (see I.1.18). We call an EARS *"simply laced"* or *"nonsimply laced"* according to whether the type of the attached finite root system is simply laced or nonsimply laced, respectively. We also call an EALA or EAWG, simply laced or nonsimply laced according to the type of the corresponding EARS. The rank and the nullity of an EARS are defined to be the rank of the finite root system attached to it and the dimension of the real span of isotropic roots respectively. We have

- any simply laced affine Lie algebra is a nontwisted affine Lie algebra, and
- any twisted affine Lie algebra is a nonsimply laced affine Lie algebra.

For a systematic study of EALA's and EARS's we refer the reader to the paper [AABGP] which, as well, is our main reference throughout this thesis. We recall many relevant results from [AABGP] in Chapter I.

This thesis consists of three chapters with Chapter I as an introduction to EALA's and EARS's. In Chapter II, we study the notion of *"twisting"* for EALA's and in Chapter III we study the structure of EAWG's. Chapters II and III can be studied independently. Chapter II, being more related to the material in Chapter I, appears right after it.

In Chapter I, we provide the basic knowledge about EALA's and EARS's. Axioms for EALA's and EARS's are given and the basic properties which we will need throughout the thesis are presented. Each statement is supplied with either a proof or a direct reference.

In Chapter II, we generalize the notion of "twisting" which appears in the realization of affine Kac-Moody algebras, to the class of EALA's, for the cases under consideration. It is well-known that twisted affine Kac-Moody algebras (which are in particular nonsimply laced EALA's) can be realized as the fixed point subalgebras of some simply laced affine Lie algebras, with respect to some diagram automorphism. Here the process of twisting can be formulated as follows.

- Describe the root systems of the Lie algebras (finite and affine).
- Assign a unique diagarm to the root system so that the vertices of the diagram are in one to one correspondence with a basis of the root system.
- Define an automorphism of a finite or affine Lie algebra which is a diagram automorphism (or graph automorphism).
- Extend the automorphism to the corresponding loop algebra.

Then the twisted affine Lie algebras are the fixed point subalgebras of these loop algebras with respect to the extended diagram automorphisms.

In 1985 the concept of (marked) EARS's was introduced by Saito [Sa1] in an attempt to construct a flat structure for the space of the universal deformation of a simple elliptic singularity. He classified, up to isomorphism, (marked) EARS's of nullity 2. Similar to the finite and affine cases he assigns a unique diagram to each EARS. Then a natural question was the realization problem, that is, are there some Lie algebras having (marked) EARS's as their root systems. Since (marked) EARS's are a natural generalization of the finite and affine root systems (see Section 1 of [Sa1]), the natural way to approach this was to give the realization by using the twisting process, this time starting from an affine Kac-Moody Lie algebra. Historically, [W] in an unpublished paper, [H-KT] and [P] tried to give realizations for the EARS's of nullity 2 by using the twisting process, where the work of [Po] completed the realization problem for EARS's of nullity 2 (see also [BR]). The basic step in the twisting process is

to realize nonsimply laced finite or affine Lie algebras as fixed point subalgebras of some simply laced finite or affine Lie algebras, respectively.

Chapter II of this thesis is devoted to investigate the process of twisting for the class of EALA's. In Chapter III of [AABGP], the authors give a construction which provides many new examples of EALA's. Modifying the construction given in [AABGP], we show that for types A_1, B, C and BC all such examples can be realized as the root systems of fixed point subalgebras of some EALA's of types A and D, with respect to some period 2 automorphism. In [AABGP] the given examples of EALA's of types A_1, B, C and BC are realized as derived subalgebras of certain subalgebras of $M_n(\mathcal{A})$ consisting of skew-symmetric elements relative to an involution of $M_n(\mathcal{A})$. Here $\mathcal{A}$ is a quantum torus. Using this, with slight modifications, it is easy to see that these examples can be obtained by the twisting process from EALA's of type A (see Section 3 and 4). We also show that (see Section 5) EALA's of type A_1 and B can be obtained by the twisting process from EALA's of type D. Here we mention that realizing a Lie algebra as a twisted subalgebra of another Lie algebra which has a simpler structure often leads to a better understanding of that Lie algebra. We also show that all the known EARS's of type BC, arising from Lie algebras, can be realized as the root systems of fixed point subalgebras of some EALA's of type C, with respect to some period 2 automorphisms. We note here that the notions of "basis" and "diagram" have only been defined for EARS's of nullity $0, 1$ and 2 (nullity 0 and 1 are just finite and affine case and for nullity 2 these notions are defined by [Sa1]). Therefore, it is interesting to see the generality of the automorphisms given in Chapter II, for each type, in the sense that the automorphisms are given in a general form which covers all the cases under consideration and that the given automorphisms do not depend on the notions of "basis" and "diagram".

In Chapter III, we study the structure of EAWG's. By definition an EAWG is a subgroup of endomorphisms of some real vector space which contains the real span of roots (see Definition III.2.15). Beside the finite Weyl groups and affine Kac-Moody Weyl groups, the toroidal Weyl groups (see [M-S]) are also EAWG's.

In 1985, Saito [Sa1] defined the Weyl groups of (marked) EARS's. Employing the Eichler-Siegel map Saito was able to study EAWG's even though at that time the structure of EARS's of nullity > 2 was not investigated in details. In 1992, Moody and Shi [M-S] studied the toroidal Weyl groups, the Weyl groups of toroidal Lie algebras. [M-S]'sapproach was based on the complete knowledge of the structure of toroidal root systems. Their work covers the Weyl groups of simply laced EARS's of rank > 1. Chapter III, in part, is a generalization of [M-S] to all EAWG's of reduced type, that is, types A, D, E, B, C, F

and G. Based on the knowledge of the structure of EARS's, investigated in [AABGP], we follow the [M-S]'s method to study the structure of an EAWG.

We start Chapter III by giving the basic definitions and results about semilattices which play an important role in the study of EAWG's. The new notions of index and duality are defined for the class of EARS's where, as it will be revealed in Chapter III, these are useful tools for the study of EAWG's and EARS's (see also [A]).

In Chapter III we show that an EAWG $\mathcal{W}$ is a semidirect product of a finite Weyl group and a characteristic subgroup H of $\mathcal{W}$, where H is a 2−step nilpotent abelian group with a center, given explicitly, which is a free abelian group of rank $\nu(\nu-1)/2$, ν being the nullity of the root system. This provides a unique expression for a given element of $\mathcal{W}$ which we will use to give a presentation for $\mathcal{W}$, when $\mathcal{W}$ has index zero, called "a presentation by conjugation". This generalizes the works of [Sh] and [Kr] for simply laced cases of rank>1. We will also consider for the class of EARS's, one of the characterizations of a basis of a finite or affine root systems. Namely, for an EARS of type X, there exists a subset $\Pi(X)$ of nonisotropic roots so that all the nonisotropic roots can be recovered by the action on $\Pi(X)$ of the subgroup $\mathcal{W}_{\Pi(X)}$ of $\mathcal{W}$ generated by reflections r_α, $\alpha \in \Pi(X)$. Moreover $\Pi(X)$ has the least cardinality $\text{ind}(R) + \ell + \nu$ with this property, where $\text{ind}(R)$ is an invariant of R, ℓ is the rank of R and ν is the nullity of the EARS R. In the finite or affine case, that is when $\nu = 0$, or 1, $\Pi(X)$, $\mathcal{W}_{\Pi(X)}$ and $\text{ind}(R) + \ell + \nu$ are, as one expects, a set of simple roots, the Weyl group, and the rank of R, respectively.

We conclude Chapter III by generalizing the results given in Section 2 of [M-S], showing that the Weyl group of an EARS of index zero is the homomorphic image of some Weyl group of indefinite type where the homomorphism and its kernel are given explicitly.

Chapter 1

Introduction to Extended Affine Lie Algebras

1 Extended Affine Lie algebras

Chapter I of this thesis, consisting of one section, is an introduction to extended affine Lie algebras (EALA) and extended affine root systems (EARS). We start off by giving the definition of an EALA $\mathcal{L}$ and recording the basic properties of $\mathcal{L}$ and its EARS R. The results of this chapter appear in [H-KT], [BGK] and [AABGP], where [AABGP] is our main reference for this chapter. We have provided a proof in the cases in which we could not find a direct reference in which the proof is given. We have recorded only those basic properties of EALA's and EARS's which are of use to us in this thesis. We refer the reader to [AABGP, I], for a through study of the basic properties of EALA's.

Definition 1.1 *An Extended affine Lie algebra (EALA for short) is a triple* $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ *consisting of two complex Lie algebra* $\mathcal{H}$ *and* $\mathcal{L}$ *with* $\mathcal{H} \subseteq \mathcal{L}$, *and a symmetric bilinear form* $(\cdot,\cdot) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ *such that the axioms (EA1) through (EA5) described below hold.*

(EA1) The form $(\cdot,\cdot)$ *is nondegenerate and invariant. (A form* $(\cdot,\cdot)$ *on* $\mathcal{L}$ *is called invariant if* $(x, [y, z]) = ([x, y], z)$ *for all* $x, y, z \in \mathcal{L}$.*)*

(EA2) $\{0\} \neq \mathcal{H}$ *is finite dimensional, abelian and self-centralizing. Moreover* $ad(h) \in End(\mathcal{L})$ *is diagonalizable for all* $h \in \mathcal{H}$.

From (EA2) we obtain the usual root space decomposition

$$\mathcal{L} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{L}_\alpha \tag{1.2}$$

where $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$. *Let*

$$R = \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_\alpha \neq \{0\}\}. \tag{1.3}$$

Then we can decompose R *as* $R = R^0 \uplus R^\times$, *where* R^0 *and* $R^\times$ *are the sets of isotropic and nonisotropic roots of* R *respectively. (The symbol* $\uplus$ *means disjoint union). So*

$$R^0 = \{\alpha \in R \mid (\alpha, \alpha) = 0\} \quad \textit{and} \quad R^\times = \{\alpha \in R \mid (\alpha, \alpha) \neq 0\}.$$

Our next axioms are

(EA3) $ad_{\mathcal{L}}(x)$ *is locally nilpotent for all* $x \in \mathcal{L}_\alpha$, $\alpha \in R^\times$.

(EA4) R *is a discrete subset of* $\mathcal{H}^*$.

(EA5) $\mathcal{L}$ *is irreducible; that is the root system* R *satisfies the following two conditions:*

(a) $R^\times$ *cannot be decomposed as a disjoint union* $R_1 \uplus R_2$, *where* R_1 *and* R_2 *are nonempty subsets of* $R^\times$ *satisfying* $(R_1, R_2) = \{0\}$,

(b) For any $\sigma \in R^0$, *there exists* $\alpha \in R^\times$ *such that* $\alpha + \sigma \in R$.

When there is no confusion, we simply call $\mathcal{L}$ *an EALA.*

We now state some of the immediate consequences of axioms (EA1)-(EA5).

From (EA2) we have $0 \in R$ and $\mathcal{L}_0 = \mathcal{H}$. From the Jacobi identity we get

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathcal{H}^*,$$

and from the invariancy of the bilinear form in (EA1) we get

$$(\mathcal{L}_\alpha, \mathcal{L}_\beta) = \{0\} \quad \text{unless } \alpha + \beta = 0. \tag{1.4}$$

In particulat the restriction of $(\cdot, \cdot)$ to $\mathcal{H}$ is nondegenerate. This allows us to define for any $\alpha \in \mathcal{H}^*$ a unique element $t_\alpha \in \mathcal{H}$ by requiring that

$$\alpha(h) = (t_\alpha, h) \quad \text{for all } \alpha, \beta \in \mathcal{H}^*. \tag{1.5}$$

We transfer the form to $\mathcal{H}^*$ by setting

$$(\alpha,\beta) := (t_\alpha, t_\beta) \quad \text{for all } \alpha,\beta \in \mathcal{H}^*. \tag{1.6}$$

Using the invariancy of $(\cdot,\cdot)$ and the nondegeneracy of $(\cdot,\cdot)$ restricted to $\mathcal{H}$, we easily get

$$[x,y] = (x,y)t_\alpha, \quad \text{for all } x \in \mathcal{L}_\alpha,\ y \in \mathcal{L}_{-\alpha}, \quad \alpha \in R. \tag{1.7}$$

Thus

$$[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = \mathbb{C}t_\alpha \quad \text{for all } \alpha \in R. \tag{1.8}$$

Axiom (EA3) allows us to construct automorphisms of the form

$$\theta_\alpha(t) := \exp(\mathrm{ad} te_\alpha)\exp(\mathrm{ad}(-t^{-1}f_\alpha))\exp(\mathrm{ad} te_\alpha), \tag{1.9}$$

where $e_\alpha \in \mathcal{L}_\alpha$, $f_\alpha \in \mathcal{L}_{-\alpha}$, $\alpha \in R^\times$, and $t \in \mathbb{C}^\times$.

Next for $\alpha \in R^\times$ we define $r_\alpha \in GL(\mathcal{H}^*)$ by

$$r_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \quad \text{for all } \beta \in \mathcal{H}^*. \tag{1.10}$$

Let $\mathcal{W} = \mathcal{W}_\mathcal{L}$ be the subgroup of $GL(\mathcal{H}^*)$ generated by the reflections r_α, $\alpha \in R^\times$. $\mathcal{W}_\mathcal{L}$ is called the Weyl group of $\mathcal{L}$. Let us also define for $\alpha \in R^\times$, the element $r_\alpha^* \in GL(\mathcal{H})$ by

$$r_\alpha^*(h) = h - \frac{2(t_\alpha,h)}{(t_\alpha,t_\alpha)}, \quad h \in \mathcal{H}.$$

Then using the isometry $\alpha \longmapsto t_\alpha$ from $\mathcal{H}^*$ onto $\mathcal{H}$, one can see that $\mathcal{W}_\mathcal{L}$ is isomorphic to the subgroup of $GL(\mathcal{H})$ generated br elements r_α^*, $\alpha \in R^\times$. It turns out that (see (1.26) and (1.27) of [AABGP, Chapter 1])

$$\theta_\alpha(t)h = r_\alpha^*(h) \quad \text{for } h \in \mathcal{H} \text{ and } t \in \mathbb{C} \setminus \{0\}.$$

In particular for $t \in \mathbb{C} \setminus \{0\}$,

$$\mathcal{W}_\mathcal{L} \cong \langle \theta_\alpha(t)|_\mathcal{H} \mid \alpha \in R^\times \rangle.$$

Using (1.9), we have the following result.

Theorem 1.11 *[AABGP, I.1.29] Let $\mathcal{L}$ satisfy (EA1)-(EA3) and $\alpha \in R^\times$. Then*

(a) For $\beta \in R$, we have $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(b) $r_\alpha(R) = R$.

(c) $\mathbb{C}\alpha \cap R = \{0, \pm\alpha\}$.

(d) $\dim\mathcal{L}_\alpha = 1$.

(e) For any $\beta \in R$ there exist two nonnegative integers u, d such that for any $n \in \mathbb{Z}$ we have $\beta + n\alpha \in R$ if and only if $-d \le n \le u$. Moreover, $d - u = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

We next want to describe the root system of an EALA.

Assume now that $\mathcal{L}$ is an EALA. From Proposition 2.1 of [AABGP, I] we have

$$(R, R^0) = \{0\}. \tag{1.12}$$

Let $\mathcal{V}$ be the real span of R. Then the form $(\cdot,\cdot)$ can be scaled so that

the form $(\cdot,\cdot)$ when restricted to $\mathcal{V}$ is real valued and positive semidefinite. (1.13)

(See Theorem 2.14 of [AABGP, I]). From now on we assume that (1.13) holds. Then it turns out that the root system R satisfies the axioms (R1)-(R8) listed in the next definition.

Definition 1.14 *Let $\mathcal{V}$ be a nontrivial finite dimensional real vector space with a positive semidefinite symmetric bilinear form $(.,.)$ and let R be a subset of $\mathcal{V}$. Let*

$$R^\times = \{\alpha \in R : (\alpha,\alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R : (\alpha,\alpha) = 0\}.$$

Then,

$$R = R^\times \uplus R^0.$$

R is called an extended affine root system (EARS for short) in $\mathcal{V}$ if R satisfies the following axioms:

(R1) $0 \in R$

(R2) $-R = R$

(R3) R spans $\mathcal{V}$

(R4) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$

(R5) R is discrete in $\mathcal{V}$

(R6) If $\alpha \in R^\times$ and $\beta \in R$, then there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\},$$

and $d - u = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$.

(R7) If $R^\times = R_1 \uplus R_2$, where $(R_1, R_2) = \{0\}$, then either R_1 or R_2 is empty.

(R8) For any $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

Two EARS's R and R' in $\mathcal{V}$ are said to be isomorphic, written $R \cong R'$, if there exists a linear bijection ϕ from $\mathcal{V}$ onto itself so that ϕ preserves the form $(\cdot, \cdot)$ up to a nonzero scalar and $\phi(R) = R'$.

Now let R be an EARS in $\mathcal{V}$ (in particular R can be the root system of an EALA). Let $\mathcal{V}^0$ be the radical of the form $(\cdot, \cdot)$. Let $\bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$ and let $^- : \mathcal{V} \to \bar{\mathcal{V}}$ be the canonical map. Since the form $(\cdot, \cdot)$ is positive semidefinite, we have

$$\mathcal{V}^0 = \{\alpha \in \mathcal{V} \mid (\alpha, \alpha) = 0\} \quad \text{and} \quad R^0 = R \cap \mathcal{V}^0.$$

Define a bilinear form $(\cdot, \cdot)$ on $\bar{\mathcal{V}}$ by letting

$$(\bar{a}, \bar{\beta}) = (\alpha, \beta) \quad \text{for all } \alpha, \beta \in \mathcal{V}.$$

The important fact here is that

$$\begin{gathered}\text{the form } (\cdot, \cdot) \text{ is positive definite on } \bar{\mathcal{V}} \text{ and that} \\ \bar{R}, \text{the image of } R \text{ under } ^- \text{ is a finite irreducible (not necessarily reduced) root system.}\end{gathered} \tag{1.15}$$

(Here we depart from [Bou] in assuming that an irreducible finite root system contains 0).

Definition 1.16 *The nullity of an EARS R is defined to be the dimension ν of $\mathcal{V}^0$. The type of R is defined to be the type X of the finite root system $\bar{R}$. The rank of R is defined to be the rank l of $\bar{R}$. Also R is called reduced if $\bar{R}$ is reduced. If R is the root system of an EALA $\mathcal{L}$, then the nullity, the type and the rank of $\mathcal{L}$ are defined to be the nullity, the type and the rank of R, respectively. We call the EALA $\mathcal{L}$ "simply laced" or "nonsimply laced" according to the type X of R. Therefore $\mathcal{L}$ is simply laced if it has one of the types $X = A$, D or E and is nonsimply laced if it has one of the types $X = B$, C, F, G or BC.*

We now lift $\bar{R}$ to a finite root system in $\mathcal{V}$ as follows.

$$\text{fix a basis } \bar{\Pi} = \{\alpha_1, \dots, \alpha_\ell\} \text{ for } \bar{R} \text{ and choose } \dot{\alpha}_i \text{ in } R \text{ so that } \bar{\dot{\alpha}}_i = \alpha_i. \tag{1.17}$$

Let $\dot{\mathcal{V}}$ be the real span of $\dot{\alpha}_1, \dots, \dot{\alpha}_\ell$. Then

$$\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$$

and $^-$ restricts to an isometry of $\dot{\mathcal{V}}$ onto $\bar{\mathcal{V}}$. Moreover, $^-$ sends the set

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \dot{\alpha} + \sigma \in R \text{ for some } \sigma \in \mathcal{V}^0\} \tag{1.18}$$

onto $\bar{R}$. Thus

$$\begin{array}{l}\dot{R} \text{ is a finite root system in } \dot{\mathcal{V}} \text{ isomorphic to } \bar{R} \text{ and} \\ \text{the restriction of the form on } \dot{\mathcal{V}} \text{ is positive definite.}\end{array} \tag{1.19}$$

Moreover, it follows that

$$\text{if } \dot{\alpha} \in \dot{R} \text{ is reduced, then } \dot{\alpha} \in R. \tag{1.20}$$

(See [AABGP, II.2.11]. Also recall that a nonzero root $\dot{\alpha}$ in a finite root system $\dot{R}$ is called reduced if $\frac{1}{2}\dot{\alpha} \notin \dot{R}$). Let $\dot{R}^\times = \dot{R} \setminus \{0\}$. We have $\dot{R}^\times = \dot{R}_{sh} \uplus \dot{R}_{lg} \uplus \dot{R}_{ex}$ where $\dot{R}_{sh}$, $\dot{R}_{lg}$ and $\dot{R}_{ex}$ are the sets of short, long and extra long roots of $\dot{R}$, respectively. We always assume that in simply laced cases each nonzero root is a short root. Thus $\dot{R}_{lg}$ and $\dot{R}_{ex}$ might be empty. This decomposition of $\dot{R}^\times$ will serve to give a decomposition $R^\times = R_{lg} \uplus R_{lg} \uplus R_{ex}$. More precisely, let

$$\begin{aligned} S &= \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{sh}\}, \\ L &= \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{lg}\} \quad \text{and} \\ E &= \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{ex}\}. \end{aligned}$$

Then

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E), \tag{1.21}$$

where $E \subseteq L \subseteq S$ and S, L and E satisfy some inter-relations which will be discussed in Construction 1.24 and Theorem 1.28 below. (We interpret $\dot{R}_{lg} + L$ or $\dot{R}_{ex} + E$ as empty sets if $\dot{R}_{lg}$ or $\dot{R}_{ex}$ are empty, respectively. The important thing in the above discription of R is that

$$S \text{ and } L \text{ are semilattices in } \mathcal{V}^0 \text{ and } E \text{ is a translated semilattice in } \mathcal{V}^0, \tag{1.22}$$

where the definitions of a semilattice and a translated semilattice are given bellow.

Definition 1.23 *A semilattice in a finite dimensional vector space $\mathcal{U}$ is a subset S of $\mathcal{U}$ satisfying*

$$(S1)\, 0 \in S, \quad (S2)\, S \pm 2S \subseteq S, \quad (S3)\, S \text{ spans } \mathcal{U}, \quad (S4)\, S \text{ is discrete in } \mathcal{U}.$$

We define the rank of S to be the dimension of $\mathcal{U}$. A translated semilattice is a subset S of $\mathcal{U}$ which satisfies (S2), (S3) and (S4).

The notion of a semilattice is a crucial notion in the discription of EARS's and extended affine Weyl groups (see Chapter 3 for the definition of an extended affine Weyl group). Semilattices are also used to give new examples of EALA's (see Chapter II and [AABGP, III]). To see more about semilattices see [AABGP, II.1]. We also consider semilattices in more details in III, showing new aspects of their importance in the theory of EALA's.

Up to this point, we have shown how to decompose any EARS as in (1.21) using a finite root system and up to 3 semilattices and translated semilattices. Conversely, we can use a finite root system and semilattices to construct EARS:

Construction 1.24 *Suppose that $\dot{R}$ is an irreducible finite root system of type X in a finite dimensional real vector space $\dot{\mathcal{V}}$ with a positive definite symmetric bilinear form $(\cdot,\cdot)$. We decompose the set $\dot{R}^\times$ of nonzero elements of $\dot{R}$ according to length as $\dot{R}^\times = \dot{R}_{sh} \uplus \dot{R}_{lg} \uplus \dot{R}_{ex}$. Let $\mathcal{V}^0$ be a finite dimensional real vector space, let $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and extend $(\cdot,\cdot)$ to $\mathcal{V}$ in such a way that $(\mathcal{V}, \mathcal{V}^0) = \{0\}$.*

(a) (The simply laced construction) Suppose that X is simply laced, i.e. $X = A_\ell (\ell \geq 1)$, $D_\ell (\ell \geq 4)$, E_6, E_7 or E_8. Suppose that S is a semilattice in $\mathcal{V}^0$. If $X \neq A_1$ suppose further that S is a lattice in $\mathcal{V}^0$. Put

$$R = R(X, S) := (S + S) \cup (\dot{R} + S).$$

(b) (The reduced nonsimply laced construction) Suppose that X is reduced and nonsimply laced, i.e. $X = B_\ell (\ell \geq 2)$, $C_\ell (\ell \geq 3)$, F_4 or G_2. Suppose that S and L are semilattices in $\mathcal{V}^0$ so that

$$L + kS \subseteq L \quad \textit{and} \quad S + L \subseteq S, \tag{1.25}$$

where k is defined to be 3 if $X = G_2$ and 2 otherwise. Further, if $X = B_\ell(\ell \geq 3)$ suppose that L is a lattice, if $X = C_\ell(\ell \geq 3)$ suppose that S is a lattice, and if $X = F_4$ or G_2 suppose that both S and L are lattices. Put

$$R = R(X, S, L) := (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L).$$

(c) (The BC_ℓ construction, $\ell \geq 2$) Suppose that $X = BC_\ell(\ell \geq 2)$. Suppose that S and L are semilattices in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$ such that $E \cap 2S = \emptyset$ and

$$L + 2S \subseteq L, \; S + L \subseteq S, \; E + 2L \subseteq E \quad \text{and} \quad L + E \subseteq L. \tag{1.26}$$

If $\ell \geq 3$, suppose further that L is a lattice. Put

$$R = R(BC_\ell, S, L, E) = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E).$$

(d) (The BC_1 construction) Suppose that $X = BC_1$. Suppose that S is a semilattice in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$ such that $E \cap 2S = \emptyset$ and

$$E + 4S \subseteq E \quad \text{and} \quad S + E \subseteq S. \tag{1.27}$$

Put

$$R = R(BC_1, S, E) := (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{ex} + E).$$

In the following two theorems, we state the main results on the structure of EARS's.

Theorem 1.28 *[AABGP, II.2.37] Let X be one of the types for a finite root system. Starting from a finite root system $\dot{R}$ of type X and up to three semilattices or translated semilattices (as indicated in the construction), Construction 1.24 produces an extended affine root system of type X. Conversely, any extended affine root system of type X is isomorphic to a root system obtained from the part of Construction 1.28 corresponding to type X.* □

Theorem 1.29 *[AABGP, II.3.1]*

(a) Suppose that X is simply laced and $R(X, S)$ and $R(X, S')$ are as in Construction 1.24(a). Then, $R(X, S) \cong R(X, S')$ iff there exists $\varphi \in GL(\mathcal{V}^0)$ so that

$$\varphi(S) = S' + \delta' \quad \text{for some } \delta' \in S'.$$

(b) Suppose that X is reduced nonsimply laced and $R(X,S,L)$ and $R(X,S',L')$ are as in Construction 1.24. Then $R(X,S,L) \cong R(X,S',L')$ iff there exists $\varphi \in GL(\mathcal{V}^0)$ so that

$$\varphi(S) = S' + \delta' \quad \textit{and} \quad \varphi(L) = L' + \lambda'$$

for some $\delta' \in S'$ and $\lambda' \in L'$. □

We saw that given an EALA $\mathcal{L}$, we have a nice description of the root system of $\mathcal{L}$. Now we come back to the Lie algebra $\mathcal{L}$ itself to see what more we can say about $\mathcal{L}$.

Let, as before $\mathcal{L}$ be an EALA of type X and nullity ν with root system R.

Definition 1.30 *The core of $\mathcal{L}$ is defined to be the subalgebra of $\mathcal{L}$ generated by root spaces $\mathcal{L}_\alpha$, α nonisotropic. We denote by $\mathcal{L}_c$ the core of $\mathcal{L}$.*

Since $\mathcal{L}_c$, as a subalgebra of $\mathcal{L}$, is generated by homogenous elements with respect to the grading $\mathcal{L} = \sum_{\alpha\in\mathcal{H}^*} \mathcal{L}_\alpha$, we get

$$\mathcal{L}_c = \sum_{\alpha\in\mathcal{H}^*} \mathcal{L}_c \cap \mathcal{L}_\alpha = \sum_{\alpha\in R} \mathcal{L}_c \cap \mathcal{L}_\alpha.$$

We have $\mathcal{L}_\alpha \subseteq \mathcal{L}_c$ for $\alpha \in R^\times$. Thus

$$\begin{gathered} \mathcal{L}_c = \textstyle\sum_{\alpha\in R}(\mathcal{L}_c)_\alpha, \quad \text{where} \\ (\mathcal{L}_c)_\alpha = \mathcal{L}_\alpha \text{ for } \alpha \in R^\times \quad \text{and} \quad (\mathcal{L}_c)_\alpha = \mathcal{L}_c \cap \mathcal{L}_\alpha \text{ for } \alpha \in R^0. \end{gathered} \tag{1.31}$$

From (1.4) and (1.7) we have $t_\alpha \in \mathcal{L}_c$ for $\alpha \in R^\times$. From this and axiom (EA5)(b), it follows easily that $t_\sigma \in \mathcal{L}_c$ for $\sigma \in R^0$. Thus

$$t_\alpha \in \mathcal{L}_c \quad \text{for all } \alpha \in R. \tag{1.32}$$

Lemma 1.33 *$\mathcal{L}_c$ is perfect.*

Proof. We only need to show that $\mathcal{L}_c \subseteq [\mathcal{L}_c, \mathcal{L}_c]$. Since $\mathcal{L}_c$ is generated by root spaces $\mathcal{L}_\alpha$, $\alpha \in R^\times$, it is enough to show that $\mathcal{L}_\alpha \subseteq [\mathcal{L}_c, \mathcal{L}_c]$ for $\alpha \in R^\times$. So let $\alpha \in R^\times$ and $x_\alpha \in \mathcal{L}_\alpha$. By (1.32), we have $t_\alpha \in \mathcal{L}_c$.Thus

$$\alpha(t_\alpha)x_\alpha = [t_\alpha, x_\alpha] \in [\mathcal{L}_c, \mathcal{L}_c].$$

Since $\alpha \in R^\times$, we have $\alpha(t_\alpha) = (\alpha, \alpha) \neq 0$. This gives $x_\alpha \in [\mathcal{L}_c, \mathcal{L}_c]$. □

Recall from (1.17) that $\dot{\mathcal{V}} = \sum_{i=1}^{\ell} \mathbf{R}\dot{\alpha}_i$ where $\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell \in R$. Also recall that $R^0 = R \cap \mathcal{V}^0$, $\mathcal{V}^0$ having dimension ν. From (1.21) we have $R^0 = S + S$ where S is a semilattice of rank ν. Then one can see that ([AABGP, II.1.11])

$$S \text{ contains a basis } \delta_1, \ldots, \delta_\nu \text{ of } \mathcal{V}^0 \text{ such that } \langle S \rangle = \textstyle\sum_{i=1}^{\nu} \mathbf{Z}\delta_i. \tag{1.34}$$

where $\langle S \rangle$ denotes the $\mathbf{Z}$-span of S in $\mathcal{V}^0$. Let

$$\dot{\mathcal{V}}_{\mathbf{C}} = \sum_{i=1}^{\ell} \mathbf{C}t_{\dot{\alpha}_i} \quad \text{and} \quad \mathcal{V}^0_{\mathbf{C}} = \sum_{i=1}^{\nu} \mathbf{C}t_{\delta_i}. \tag{1.35}$$

Since $\delta_1, \ldots, \delta_\nu$ are $\mathbf{R}$-linearly independent, we have

$$[\frac{\nu}{2}] \leq \dim \mathcal{V}^0_{\mathbf{C}} \leq \nu. \tag{1.36}$$

Since $(\cdot,\cdot)$ is real valued and is nondegenerate on $\dot{\mathcal{V}}$, it follows that

$$\text{the restriction of the form on } \dot{\mathcal{V}}_{\mathbf{C}} \text{ is also nondegenerate.} \tag{1.37}$$

For our later use we need to make the following definition here.

Definition 1.38 *The EALA $\mathcal{L}$ is called nondegenerate if $\dim\mathcal{V}^0_{\mathbf{C}} = \nu$. $\mathcal{L}$ is called degenerate otherwise. (The relations between degenerate and nondegenerate EALA's is clarified in a forthcoming paper by Yun Gao.)*

We continue our study of EALA $\mathcal{L}$ (without assuming that $\mathcal{L}$ is necessarily nondegenerate at this stage).

Lemma 1.39 $\mathcal{H} \cap \mathcal{L}_c = \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}$.

Proof. We have $t_{\dot{\alpha}_1}, \ldots, t_{\dot{\alpha}_\ell}, t_{\delta_1}, \ldots, t_{\delta_\nu}$ are in $\mathcal{H}$. Then it becomes clear from (1.32) that

$$\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \subseteq \mathcal{H} \cap \mathcal{L}_c.$$

We now show the reverse inclusion. First note that if $\alpha \in R (\subseteq \dot{\mathcal{V}} \oplus \mathcal{V}^0)$, then α is in the real span of $\dot{\alpha}_i$'s and δ_i's. Thus t_α is in the real span of $t_{\dot{\alpha}_i}$'s and t_{δ_i}'s. Therefore

$$t_\alpha \in \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \quad \text{for } \alpha \in R. \tag{1.40}$$

Now since $\mathcal{L}_c$ is generated by $\mathcal{L}_\alpha$, $\alpha \in R^\times$, we have

$$\mathcal{L}_c \cap \mathcal{H} = \mathcal{L}_c \cap \mathcal{L}_0 = \sum_{\alpha \in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}].$$

Thus from (1.8) and (1.40) we get

$$\mathcal{L}_c \cap \mathcal{H} = \sum_{\alpha \in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subseteq \sum_{\alpha \in R^\times} \mathbb{C} t_\alpha \subseteq \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}.$$

This completes the proof. □

Next let

$$\mathcal{L}_c^\perp = \{x \in \mathcal{L} \mid (x, \mathcal{L}_c) = \{0\}\}, \tag{1.41}$$

the orthogonal complement of $\mathcal{L}_c$ in $\mathcal{L}$ with respect to $(\cdot,\cdot)$. Also let

$$C_{\mathcal{L}}(\mathcal{L}_c) = \{x \in \mathcal{L} \mid [x, \mathcal{L}_c] = \{0\}\},$$

the centralizer of $\mathcal{L}_c$ in $\mathcal{L}$. By [BGK, 3.6], we have

$$\mathcal{L}_c^\perp = C_{\mathcal{L}}(\mathcal{L}_c). \tag{1.42}$$

Thus $\mathcal{Z}(\mathcal{L}_c) \subseteq \mathcal{L}_c^\perp$, where $\mathcal{Z}(\mathcal{L}_c)$ denotes the center of $\mathcal{L}_c$.

Lemma 1.43 $\mathcal{Z}(\mathcal{L}_c) \subseteq \sum_{\sigma \in R^0} \mathcal{L}_\sigma$.

Proof. Let $x \in \mathcal{Z}(\mathcal{L}_c) = \mathcal{L}_c^\perp$. Then $(x, \mathcal{L}_c) = \{0\}$. In particular $(x, \mathcal{L}_\alpha) = \{0\}$ for all $\alpha \in R^\times$. Then from the nondegeneracy of the form and (1.4) it follows that $x \in \sum_{\alpha \in R^0} \mathcal{L}_\alpha$. □

Lemma 1.44 $\mathcal{Z}(\mathcal{L}_c) \cap \mathcal{H} = \mathcal{V}^0_{\mathbf{C}}$.

Proof. We have $\mathcal{Z}(\mathcal{L}_c) \subseteq \mathcal{L}_c^\perp$. Thus $(\mathcal{Z}(\mathcal{L}_c) \cap \mathcal{H}, \mathcal{L}_c \cap \mathcal{H}) = \{0\}$. Since $\mathcal{Z}(\mathcal{L}_c) \cap \mathcal{H} \subseteq \mathcal{L}_c \cap \mathcal{H} = \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}$, by lemma 1.39, we get $\mathcal{Z}(\mathcal{L}_c) \cap \mathcal{H} \subseteq \mathcal{V}^0_{\mathbf{C}}$. To get equality, we only need to show that $t_{\delta_1}, \ldots, t_{\delta_\nu} \in \mathcal{Z}(\mathcal{L}_c)$. But for $\alpha \in R$ and $1 \leq i \leq \nu$,

$$[t_{\delta_i}, \mathcal{L}_\alpha] = \alpha(t_{\delta_i})\mathcal{L}_\alpha = (\alpha, \delta_i)\mathcal{L}_\alpha = \{0\},$$

since δ_i is isotropic. Thus $t_{\delta_i} \in \mathcal{Z}(\mathcal{L}_c)$. □

From (1.32) we have $t_\alpha \in \mathcal{L}_c$ for any $\alpha \in R$, thus

$$(t_\alpha, \mathcal{Z}(\mathcal{L}_c)) \subseteq (t_\alpha, \mathcal{L}_c^\perp) = \{0\} \quad \text{for all } \alpha \in R. \tag{1.45}$$

Lemma 1.46 *Let $\alpha \in R$, then*

$$t_\alpha \in \mathcal{Z}(\mathcal{L}_c) \text{ if and only if } \alpha(\mathcal{H} \cap \mathcal{L}_c) = \{0\}.$$

Proof. From Lemmas 1.44 and 1.39 we have

$$t_\alpha \in \mathcal{Z}(\mathcal{L}_c) \Leftrightarrow t_\alpha \in \mathcal{Z}(\mathcal{L}_c) \cap \mathcal{H} \Leftrightarrow t_\alpha \in \mathcal{V}^0_{\mathbf{C}} \Leftrightarrow (t_\alpha, \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}) = \{0\} \Leftrightarrow \alpha(\mathcal{H} \cap \mathcal{L}_c) = \{0\}.$$

□

If $\alpha \in R^\times$ and $\sigma \in R^0$, then $\alpha + \sigma$ is nonisotropic and so

$$[\mathcal{L}_\alpha, \mathcal{L}_\sigma] \subseteq \mathcal{L}_{\alpha+\sigma} \subseteq \mathcal{L}_c.$$

From this, it follows that

$$\mathcal{L}_c \text{ is an ideal of } \mathcal{L}. \tag{1.47}$$

Hence we have a representation

$$\rho : \mathcal{L} \longrightarrow Der(\mathcal{L}_c), \quad \text{(the derivation algebra of } \mathcal{L}_c)$$

given by

$$\rho(x)(y) = [x, y], \quad \text{for } x \in \mathcal{L},\ y \in \mathcal{L}_c.$$

Clearly the kernel of ρ is just $C_{\mathcal{L}}(\mathcal{L}_c)$ which is equal to $\mathcal{L}_c^\perp$ by (1.42).

Definition 1.48 *The EALA $\mathcal{L}$ is called tame if $\mathcal{L}_c^\perp$ equals the center $\mathcal{Z}(\mathcal{L}_c)$ of the core. By (1.42), we have $\mathcal{L}$ is tame if and only if $\mathcal{L}_c^\perp \subseteq \mathcal{L}_c$.*

From now on we assume that $\mathcal{L}$ is a tame EALA. Since the restriction of the form to both $\mathcal{H}$ and $\dot{\mathcal{V}}_{\mathbf{C}}$ is nondegenerate and $(\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}, \mathcal{V}^0_{\mathbf{C}}) = \{0\}$, it follows that there exists a subspace $\mathcal{D}$ of $\mathcal{H}$ which satisfies

$$\begin{aligned} &\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \oplus \mathcal{D} \subseteq \mathcal{H}, \\ &\dim \mathcal{D} = \dim \mathcal{V}^0_{\mathbf{C}} \text{ and} \\ &(\dot{\mathcal{V}}_{\mathbf{C}}, \mathcal{D}) = \{0\}. \end{aligned}$$

By Lemma 1.39, we have

$$\mathcal{D} \cap \mathcal{L}_c = \{0\}. \tag{1.49}$$

Lemma 1.50 $\mathcal{H} = \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \oplus \mathcal{D}$.

Proof. Let $\mathcal{W}'$ be a complement of $\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \oplus \mathcal{D}$ in $\mathcal{H}$ such that $(\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}, \mathcal{W}') = \{0\}$. By Lemma 1.39, $\mathcal{L}_c \cap \mathcal{H} = \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}}$ and $\mathcal{L}_c \cap \mathcal{W}' = \{0\}$. Thus

$$(\mathcal{L}_c, \mathcal{W}') = (\sum_{\alpha \in R} \mathcal{L}_c \cap \mathcal{L}_\alpha, \mathcal{W}') = (\mathcal{L}_c \cap \mathcal{L}_0, \mathcal{W}') = \{0\}.$$

Therefore $\mathcal{W}' \subseteq \mathcal{L}_c^\perp = \mathcal{Z}(\mathcal{L}_c) \subseteq \mathcal{L}_c$, since $\mathcal{L}$ is tame. Hence $\mathcal{W}' = \{0\}$. □

By (1.36), $[\frac{\nu}{2}] \leq \dim \mathcal{V}^0_{\mathbf{C}} \leq \nu$. Let $\mu = \dim \mathcal{V}^0_{\mathbf{C}}$. Without loss of generality, we can assume that $t_{\delta_1}, \ldots, t_{\delta_\mu}$ are $\mathbf{C}$-linearly independent. Since $(\dot{\mathcal{V}}_{\mathbf{C}}, \mathcal{V}^0_{\mathbf{C}}) = \{0\}$, $(\mathcal{V}^0_{\mathbf{C}}, \mathcal{V}^0_{\mathbf{C}}) = \{0\}$ and the restriction of the form to $\mathcal{H}$ is nondegenerate, it follows from Lemma 1.50 that

$$\begin{gathered}\text{there exists a basis } d_1, \ldots, d_\mu \text{ of } \mathcal{D} \text{ such that} \\ (t_{\delta_i}, d_j) = \delta_{ij}, \quad 1 \leq i, j \leq \mu.\end{gathered} \tag{1.51}$$

Let Λ_i be the unique element in $\mathcal{H}^*$ so that $t_{\Lambda_i} = d_i$, $1 \leq i \leq \mu$. Then $\mathcal{H}^* = \sum_{i=1}^{\ell} \mathbf{C}\alpha_i \oplus \sum_{i=1}^{\mu} \mathbf{C}\delta_i \oplus \sum_{i=1}^{\mu} \mathbf{C}\Lambda_i$. We set

$$\tilde{\mathcal{V}} := \sum_{i=1}^{\ell} \mathbf{R}\alpha_i \oplus \sum_{i=1}^{\mu} \mathbf{R}\delta_i \oplus \sum_{i=1}^{\mu} \mathbf{R}\Lambda_i.$$

Then $R \subseteq \tilde{\mathcal{V}} \subseteq \mathcal{H}^*$; and the form restricted to $\tilde{\mathcal{V}}$ is nondegenerate. For $\alpha \in R^\times$ we define the reflection $r_\alpha \in GL(\tilde{\mathcal{V}})$ by $r_\alpha(\lambda) = \lambda - (\lambda, \check{\alpha})\alpha$; and we form the group

$$\mathcal{W}_\mu = \langle r_\alpha \mid \alpha \in R^\times \rangle \subseteq GL(\tilde{\mathcal{V}}). \tag{1.52}$$

Recall the Weyl group $\mathcal{W}_{\mathcal{L}}$ of the EALA $\mathcal{L}$ defined earlier in this section. It is easy to see that

$$\mathcal{W}_{\mathcal{L}} \cong \mathcal{W}_\mu. \tag{1.53}$$

Indeed the assignment $r_\alpha \longmapsto r_{\alpha|_{\tilde{\mathcal{V}}}}$, $\alpha \in R^\times$ induces this isomorphism. In Chapter III, we investigate structure of the Weyl group $\mathcal{W}_\mu$.

Next, we want to make $\mathcal{L}$ into a $\mathbf{Z}^\nu$-graded Lie algebra. For this purpose

$$\text{we assume, from now on, that } \mathcal{L} \text{ is a nondegenerate EALA.} \tag{1.54}$$

(See Definition 1.38.) By (1.51)

$$\begin{gathered}\text{that there exists a basis } d_1, \ldots, d_\nu \text{ of } \mathcal{D} \text{ such that} \\ (t_{\delta_i}, d_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq \nu.\end{gathered}$$

Note that by (1.21) and (1.34), any $\alpha \in R$ can be written in the form $\alpha = \sum_{i=1}^{\ell} n_i\dot{\alpha}_i + \sum_{j=1}^{\nu} m_j\delta_j$, $n_i, m_j \in \mathbf{Z}$. Then for $x \in \mathcal{L}_\alpha$, we have

$$[d_j, x] = \alpha(d_j)x = m_j x, \quad j = 1, \dots, \nu.$$

That is d_j is the j−th degree derivation. We define

$$deg(x) = (m_1, \dots, m_\nu) \in \mathbf{Z}^\nu \quad \text{for } x \in \mathcal{L}_\alpha, \ \alpha = \sum_{i=1}^{\ell} n_i\dot{\alpha}_i + \sum_{j=1}^{\nu} m_j\delta_j \in R. \tag{1.55}$$

This defines a grading on $\mathcal{L}$ so that

$$\mathcal{L} = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}^\sigma \quad \text{where} \quad \mathcal{L}^\sigma = \{x \in \mathcal{L} \mid deg(x) = \sigma\}.$$

Lemma 1.56 *Let $\mathcal{G}$ be a Lie algebra and M an abelian subalgebra of $\mathcal{G}$ with a decomposition $\mathcal{G} = \bigoplus \sum_{\alpha \in M^*} \mathcal{G}_\alpha$ where*

$$\mathcal{G}_\alpha = \{x \in \mathcal{G} \mid [h, x] = \alpha(h)x, \ \textit{for all } \alpha \in M^*\}.$$

Then for any grading $\mathcal{G} = \sum_{a \in \mathcal{A}} \mathcal{D}^a$ of $\mathcal{G}$ with $M \subseteq \mathcal{D}^0$ and $\mathcal{A}$ and abelian group, we have

$$\mathcal{G}_\alpha = \sum_{a \in \mathcal{A}} \mathcal{G}_\alpha \cap \mathcal{D}^a, \quad \alpha \in M^*.$$

Proof. The inclusion "$\supseteq$" is clear. To see the reverse inclusion let $\alpha \in M^*$ and $x \in \mathcal{G}_\alpha \subseteq \mathcal{G} = \sum_{a \in A} D^a$. Then

$$x = x^{a_1} + \cdots + x^{a_t} \quad \text{for some } x^{a_i} \in D^{a_i}, \quad a_i \neq a_j \text{ if } i \neq j.$$

Therefore, it is enough to show that $x^{a_i} \in \mathcal{L}_\alpha$, for $i = 1, \dots, t$. For $h \in M$ we have

$$\sum_{i=1}^{t} \alpha(h)x^{a_i} = \alpha(h)x = [h, x] = \sum_{i=1}^{t} [h, x^{a_i}], \quad \text{where}$$

$$\alpha(h)x^{a_i} \in D^{a_i} \quad \text{and} \quad [h, x^{a_i}] \in [M, D^{a_i}] \subseteq [D^0, D^{a_i}] \subseteq D^{a_i}.$$

Thus $\alpha(h)x^{a_i} = [h, x^{a_i}]$ for all $h \in M$. Hence $x^{a_i} \in \mathcal{G}_\alpha$. □

Since $\mathcal{L}_0 = \mathcal{H} \subseteq \mathcal{L}^0$, Lemma 1.56 gives

$$\mathcal{L}_\alpha = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}_\alpha \cap \mathcal{L}^\sigma \quad \text{for } \alpha \in R. \tag{1.57}$$

Since $\mathcal{L}_c$ is generated by $\mathcal{L}_\alpha$, $\alpha \in R^\times$, and $\mathcal{L}_\alpha$ is generated by homogenous elements with respect to the $\mathbf{Z}^\nu$-grading on $\mathcal{L}$, we get that $\mathcal{L}_c$ is $\mathbf{Z}^\nu$-graded with

$$\mathcal{L}_c = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}_c^\sigma \quad \text{where} \quad \mathcal{L}_c^\sigma = \mathcal{L}_c \cap \mathcal{L}^\sigma, \;\; \sigma \in \mathbf{Z}^\nu. \tag{1.58}$$

Thus from (1.57), (1.31) and (1.58) we have

$$\mathcal{L} = \sum_{\alpha \in R} \mathcal{L}_\alpha = \sum_{\alpha \in R} \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}^\sigma \cap \mathcal{L}_\alpha \quad \text{and} \tag{1.59}$$

$$\mathcal{L}_c = \sum_{\alpha \in R} \mathcal{L}_c \cap \mathcal{L}_\alpha = \sum_{\alpha \in R} \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}^\sigma \cap \mathcal{L}_\alpha \cap \mathcal{L}_c. \tag{1.60}$$

Chapter 2

Twisting by Automorphisms

Introduction

The objective of this chapter is to show that, as in the case of affine Kac-Moody Lie algebras, some of the known EALA's can be realized as the fixed point subalgebras of some other EALA's by a twisting process. The process of twisting is a well-known phenomena which first appeared in the attempt to find realizations for twisted affine Lie algebras starting from a nontwisted one.

One of the most important points in the development of the theory of Kac-Moody Lie algebras is to find a realization of such Lie algebras. The class of affine Lie algebras is divided into the subclasses of "twisted" and "nontwisted" affine Lie algebras (see [Ka], [M-P]). On the other hand we can also divide the class of affine Lie algebras into the classes of "simply laced" and "nonsimply laced" affine Lie algebras (see Definition I.1.15). Then we have

- any simply laced affine Lie algebra is a nontwisted affine Lie algebra, and
- any twisted affine Lie algebra is a nonsimply laced affine Lie algebra.

Here we note that a Lie algebra over the complex field is an affine Kac-Moody Lie algebra if and only if it is a tame EALA with nullity $\nu = 1$ (see [ABGP]).

It is well known that nontwisted (in particular simply laced) affine Lie algebras can be realized as the tensor product over $\mathbb{C}$ of a simple finite dimensional complex Lie algebra with the Laurent polynomials in one variable. We must make clear that by an affine Lie algebra here, we simply mean the loop version, that is no central element or derivation is added. The nonsimply laced (in particular twisted) affine Lie algebras can be realized as the fixed point subalgebras of simply laced affine Lie algebras with respect to some finite order automorphism. The process in part shows that all finite dimensional complex Lie algebras of nonsimply laced type can be realized as the fixed point subalgebras of simply laced ones, with respect to some diagram automorphism. We will refer to this pattern, that is the pattern of realizing an algebra as the fixed point subalgebra of some other algebra, as the *"twisting pattern"*.

In the development of the theory of EALA's, a basic natural question is the realization problem. Namely, for a given type and fixed nullity, axioms of an EALA $\mathcal{L}$, allow the description of all the possible sets R which can be considered as the root system of $\mathcal{L}$. Now the realization problem is that given such a root system R, is there any EALA $\mathcal{L}$ which has R as its root system.

In 1990, Høegh-Krohn and B. Torresani, [H-KT] introduced axioms for a very interesting class of Lie algebras which they called *"quasi simple Lie algebras"*. (In [AABGP], after some modification and simplification of axioms this class is called EALA's.) [H-KT] describe the corresponding root systems and give a partial answer to the realization problem. (The description of root systems given in [H-KT] contains some inaccuracies.) Basically, the realization for a simply laced EALA of type X and nullity ν is the generalized loop algebra $\dot{\mathcal{G}} \otimes_{\mathbb{C}} \mathbb{C}[x_1^{\pm 1}, \ldots, x_\nu^{\pm 1}]$, where $\dot{\mathcal{G}}$ is a simple finite dimensional Lie algebra of type X and $\mathbb{C}[x_1^{\pm 1}, \ldots, x_\nu^{\pm 1}]$ is the commutative ring of Laurent polynomials in ν variables (see [H-KT], [BGK], [AABGP]). For the construction of EALA's of nonsimply laced type, [H-KT] follow the twisting pattern. In general if the root system R is nonsimply laced with nullity $\nu = 2$, they realize the EALA with root system R as the fixed point subalgebra of some simply laced EALA of nullity 2 with respect to some finite order automorphism.

The root systems of the examples given in [H-KT] do not cover all the possible EARS's of nullity 2. Later in 1994, U. Pollmann [Po] in her thesis followed exactly the same method, the twisting pattern, to give realizations for EALA's of nullity $\nu = 2$. Her work completed the realization problem for the case $\nu = 2$.

In 1985 Saito [Sai] classified all (marked) EARS's of nullity $\nu = 2$. He assigns a diagram to each such root systems which determines the root system uniquely. He shows that each (marked) EARS of nonsimply laced type can be realized as the fixed point subset of some diagram automorphism of a (marked) EARS of simply laced type.

We would like to remark here that there are plenty of EARS's which do not arise as the root systems of known examples of EALA's. In fact in [AABGP], the authors conjectured that

there are EARS of type F_4, G_2, B_2, $C_\ell(\ell \geq 3)$ and $BC_\ell(\ell \geq 1)$
which do not arise as the root system of an EALA.

(We learned in a private communication from B. Allison and Y. Gao that this conjecture is now known to be true for the cases B_2, $C_\ell(\ell \geq 3)$, F_4 and G_2.) We formulate the following question which arises naturally with respect to the above twisting pattern.

Can nonsimply laced EARS's be realized as the root systems of fixed point subalgebras of some simply laced EALA's, with respect to some finite order automorphisms?.

We will give a positive answer to this question for the types B, C and BC. In Chapter III of [AABGP], the authors give a construction which provides the most general form of the known examples of EALA's, in the sense that the root systems of such examples covers all the known EARS's, arising from Lie algebras, given in [AABGP] or given by others. Modifying the construction given in [AABGP], we show that for types A_1, B, C and BC all the known EARS's, arising from algebras, can be realized as the root systems of fixed point subalgebras of some EALA's of types A and D, with respect to some period 2 automorphism. In [AABGP] the given examples of EALA's of types A_1, B, C and BC are realized as derived subalgebras of certain subalgebras of $M_n(\mathcal{A})$ consisting of skew-symmetric elements relative to an involution of $M_n(\mathcal{A})$. Here $\mathcal{A}$ is a quantum torus. Using this, with slight modifications, it is easy to see that these examples can be obtained by the twisting process from EALA's of type A (see Sections 3 and 4). We also show that (see Section 5) EALA's of type A_1 and B can be obtained by the twisting process from EALA's of type D. Here we mention that realizing a Lie algebra as a twisted subalgebra of another Lie algebra which has a simpler structure often leads to a better understanding of that Lie algebra. We also show that all the known EARS's of type BC, arising from Lie algebras, can be realized as the root systems of fixed point subalgebras of some EALA's of type C, with respect to some period 2 automorphisms. An interesting point is the generality of the automorphisms which we give in each type, in the sense that the automorphism is given by a general matrix which covers all the cases under consideration and that the given automorphisms are independent of the notion of a diagram. We remark here that all of our examples of Lie algebras will be Lie subalgebras of some matrix algebra $M_n(\mathcal{A})$ with commutator product in which $\mathcal{A}$ is a *quantum torus*. Here the only quantum tori which we need are those which are associative algebras generated by a finite number of generators X_i, X_i^{-1} which commute modulo ± 1.

1 A General Construction of Extended Affine Lie Algebras

In this section, we start by constructing an EALA $\mathcal{L}$ of nullity ν from a given Lie algebra $\mathcal{G}$ which we call a (tame) generalized loop algebra (see Definition 1.20), satisfying some prescribed conditions. The imposed conditions on $\mathcal{G}$ are a modified version of those given in [AABGP, III.1]. The objective is that our constructed EALA $\mathcal{L}$ covers all the examples which will arise naturaly in future sections while some of these examples do not fit the construction given in [AABGP, III.1].

As it is shown in [AABGP,III.1.20] when $\mathcal{G}$ is tame, the constructed EALA $\mathcal{L}$ is also tame. Moreover $\mathcal{G}$ is a central quotient of the core of $\mathcal{L}$. In Proposition 1.35 we give a converse result to this, namely, given any (nondegenerate) tame EALA $\mathcal{L}$, the quotient algebra of the core of $\mathcal{L}$ modulo its center is a tame generalized loop algebra. We also state some results regarding the core of $\mathcal{G}$ (see Definition 1.20).

We start with a Lie algebra $\mathcal{G}$ satisfying 9 conditions, (C1)-(C9) bellow. This will let us construct a Lie algebra $\mathcal{L}$ satisfying axioms (EA1)-(EA4) of an EALA. Imposing two more conditions, (C10)-(C11), on $\mathcal{G}$, $\mathcal{L}$ becomes an EALA which might not be tame. Imposing one more condition, (C12), $\mathcal{L}$ becomes tame. We refer the reader to [AABGP, III.1] for more details on this section.

Let ν be a positive integer. Let $(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}})$ be a triple consisting of two complex Lie algebra $\dot{\mathcal{H}}$ and $\mathcal{G}$ ($\dot{\mathcal{H}} \subseteq \mathcal{G}$) and a symmetric bilinear form $(\cdot,\cdot) : \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{C}$, satisfying conditions (C1)-(C9) bellow.

(C1) The form $(\cdot,\cdot)$ is nondegenerate and invariant.

(C2) $\dot{\mathcal{H}}$ is a nontrivial finite dimensional abelian subalgebra of $\mathcal{G}$ such that $ad_{\mathcal{G}}(\dot{\mathcal{H}})$ is diagonalizable, and

(C3) $(\cdot,\cdot)|_{\dot{\mathcal{H}}\times\dot{\mathcal{H}}}$ is nondegenerate.

Because of (C3), we may transfer $(\cdot,\cdot)$ to a form on $\dot{\mathcal{H}}^*$, the (complex) dual space of $\dot{\mathcal{H}}$. Also, by (C2), we have

$$\mathcal{G} = \sum_{\dot{\alpha}\in\dot{\mathcal{H}}^*} \mathcal{G}_{\dot{\alpha}}, \quad \text{where} \quad \mathcal{G}_{\dot{\alpha}} = \{x \in \mathcal{G} : [h, x] = \dot{\alpha}(h)x \text{ for all } h \in \dot{\mathcal{H}}\}.$$

Put

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{H}}^* : \mathcal{G}_{\dot{\alpha}} \neq \{0\}\} \quad \text{and} \quad \dot{R}^\times = \dot{R} \setminus \{0\}. \tag{1.1}$$

Assume next that

(C4) The restriction of the form $(\cdot,\cdot)$ to the real space $\dot{\mathcal{V}}$ spanned by $\dot{R}$ is a positive definite real valued form such that $\dot{R}$ is an irreducible finite root system in $\dot{\mathcal{V}}$ and

(C5) $\mathcal{G} = \oplus \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{G}^\sigma$ is $\mathbf{Z}^\nu$-graded as a Lie algebra.

(Recall that an algebra $\mathcal{Y}$ with product $\cdot$ is said to be $\mathbf{Z}^\nu$*-graded* if $\mathcal{Y} = \oplus \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{Y}^\sigma$, where $\mathcal{Y}^\sigma$, $\sigma \in \mathbf{Z}^\nu$, are subspaces of $\mathcal{Y}$ such that $\mathcal{Y}^\sigma \cdot \mathcal{Y}^\tau \subseteq \mathcal{Y}^{\sigma+\tau}$ for $\sigma, \tau \in \mathbf{Z}^\nu$. In that case, if $y \in \mathcal{Y}^\sigma$, we say that y has *degree* σ and write $deg(y) = \sigma$.)

We assume further that the $\mathbf{Z}^\nu$-grading on $\mathcal{G}$ in (C5) has the following compatibility properties:

(C6) $\mathcal{G}_{\dot{\alpha}} = \sum_{\sigma \in \mathbf{Z}^\nu} (\mathcal{G}^\sigma \cap \mathcal{G}_{\dot{\alpha}})$ for $\dot{\alpha} \in \dot{R}$,

(C7) $\dot{\mathcal{H}} = \mathcal{G}^0 \cap \mathcal{G}_0$,

(C8) $\{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}^\sigma \neq \{0\}\}$ generates a subgroup of $\mathbf{Z}^\nu$ of rank ν, and

(C9) $\sigma, \tau \in \mathbf{Z}^\nu, \sigma + \tau \neq 0 \Longrightarrow (\mathcal{G}^\sigma, \mathcal{G}^\tau) = \{0\}$.

Before introducing some more conditions on $\mathcal{G}$ we will derive some implications from (C1)-(C9).

It follows from (C5) that we may define $d_i \in Der(\mathcal{G})$, for $i = 1, \ldots, \nu$, by

$$d_i x = n_i x$$

for $x \in \mathcal{G}^{(n_1, \cdots, n_\nu)}$. Also it follows from (C8) and (C9) that

$$\begin{aligned}
&(a) \quad d_1, \ldots, d_\nu \text{ commute and are independent over } \mathbf{C}, \\
&(b) \quad (d_i x, y) = -(x, d_i y) \quad \text{for all } x, y \in \mathcal{G}, \ \ 1 \leq i \leq \nu, \\
&(c) \quad (d_i[x,y], z) + (d_i[y,z], x) + (d_i[z,x], y) = 0 \quad \text{for all } x, y, z \in \mathcal{G}, \ \ 1 \leq i \leq \nu.
\end{aligned} \tag{1.2}$$

(For details see [AABGP, III.1.12]).

We now define a triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ which will finally give our desired EALA. We start with $\mathcal{L}$. Let

$$\mathcal{L} = \mathcal{G} \oplus \mathcal{C} \oplus \mathcal{D}. \tag{1.3}$$

where

$$\mathcal{C} = \mathbf{C}c_1 \oplus \cdots \oplus \mathbf{C}c_\nu, \quad \text{a } \nu\text{-dimensional vector space, and}$$
$$\mathcal{D} = \mathbf{C}d_1 \oplus \cdots \oplus \mathbf{C}d_\nu \subseteq Der(\mathcal{G}).$$

Define the anti-commutative product $[\cdot,\cdot]'$ on $\mathcal{L}$ as follows:

$$[\mathcal{L},\mathcal{C}]' = \{0\}, [\mathcal{D},\mathcal{D}]' = \{0\},$$
$$[d_i, x]' = d_i x \quad \text{for all } x \in \mathcal{G}, \text{ and}$$
$$[x,y]' = [x,y] + \textstyle\sum_{i=1}^{\nu}(d_i x, y)c_i \quad \text{for all } x, y \in \mathcal{G}.$$

It follows easily from (1.2)(c) that $\mathcal{L}$ is a Lie algebra over $\mathbf{C}$. Next, define the form $(\cdot,\cdot)$ on $\mathcal{L}$ such that

- $(\cdot,\cdot)$ extends the form $(\cdot,\cdot)$ on $\mathcal{G}$,
- $(\mathcal{C},\mathcal{C}) = (\mathcal{D},\mathcal{D}) = \{0\}$,
- $(c_i, d_j) = \delta_{ij},\ i,j = 1,\ldots,\nu$, and
- $(\mathcal{C},\mathcal{G}) = (\mathcal{D},\mathcal{G}) = \{0\}$.

It follows from (C1) and the definition of $[\cdot,\cdot]'$ that

$$\text{the extended form } (\cdot,\cdot) \text{ on } \mathcal{L} \text{ is nondegenerate and invariant.} \tag{1.4}$$

Next put

$$\mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{C} \oplus \mathcal{D}.$$

By (C2) and (C7), $\mathcal{H}$ is an abelian subalgebra of $\mathcal{L}$. We can identify

$$\mathcal{H}^* = \dot{\mathcal{H}}^* \oplus \mathcal{C}^* \oplus \mathcal{D}^*.$$

Let $\{\delta_1,\ldots,\delta_\nu\}$ be the basis for $\mathcal{D}^*$ dual to $\{d_1,\ldots,d_\nu\}$. Identify $\mathbf{Z}^\nu$ as a subset of $\mathcal{D}^*$ through

$$(n_1,\ldots,n_\nu) = \sum_{i=1}^{\nu} n_i\delta_i.$$

Then,

$$[d,x] = \sigma(d)x \quad \text{for } d \in \mathcal{D}, x \in \mathcal{G}^\sigma \text{ and } \sigma \in \mathbf{Z}^\nu. \tag{1.5}$$

We transfer the form $(\cdot,\cdot)$ on $\mathcal{H}$ to $\mathcal{H}^*$ using the elements t_α, $\alpha \in \mathcal{H}^*$ (see (I.1.6)). Let $\{\gamma_1,\ldots\gamma_\nu\}$ be the basis for $\mathcal{C}^*$ dual to $\{c_1,\ldots,c_\nu\}$. It is clear from definition of t_α's that

$$t_{\dot{\alpha}} \in \dot{\mathcal{H}}, \quad t_{\delta_i} = c_i \quad \text{and} \quad t_{\gamma_i} = d_i, \tag{1.6}$$

for $\dot{\alpha} \in \dot{\mathcal{H}}^*$ and $i = 1, \ldots, \nu$. Hence if we restrict the form on $\mathcal{H}^*$ to $\dot{\mathcal{H}}^*$, we get the form transferred from $\dot{\mathcal{H}}$ at the beginning of this section, and

$$(\dot{\mathcal{H}}^*, \mathcal{D}^* \oplus \mathcal{C}^*) = \{0\}, \quad (\mathcal{D}^*, \mathcal{D}^*) = \{0\}, \quad (\mathcal{C}^*, \mathcal{C}^*) = \{0\} \text{ and } (\gamma_i, \delta_j) = \delta_{ij} \tag{1.7}$$

for $1 \leq i, j \leq \nu$.

For $\alpha \in \mathcal{H}^*$, let

$$\mathcal{L}_\alpha = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}.$$

Then, by (C2), (C5), (C6), (C7) and (1.5), we have

$$\mathcal{L} = \sum_{\alpha \in \mathcal{H}^*} \mathcal{L}_\alpha = \sum_{\sigma \in \mathbf{Z}^\nu} \sum_{\dot{\alpha} \in \dot{R}} \mathcal{L}_{\dot{\alpha}+\sigma}, \tag{1.8}$$

where

$$\mathcal{L}_0 = (\mathcal{G}_0 \cap \mathcal{G}^0) \oplus \mathcal{C} \oplus \mathcal{D} = \dot{\mathcal{H}} \oplus \mathcal{C} \oplus \mathcal{D} = \mathcal{H}, \tag{1.9}$$

and

$$\mathcal{L}_{\dot{\alpha}+\sigma} = \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma, \quad \text{for } \dot{\alpha} \in \dot{R},\ \sigma \in \mathbf{Z}^\nu \text{ with } \dot{\alpha} + \sigma \neq 0. \tag{1.10}$$

Therefore, if we put

$$R = \{\alpha \in \mathcal{H}^* : \mathcal{L}_\alpha \neq \{0\}\},$$

we get

$$R = \{\dot{\alpha} + \sigma \mid \dot{\alpha} \in \dot{R}, \sigma \in \mathbf{Z}^\nu, \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \neq \{0\}\} \subseteq \dot{R} + \mathbf{Z}^\nu. \tag{1.11}$$

Note that, by (C4),

$$(\dot{\alpha}, \dot{\alpha}) > 0 \quad \text{for } \dot{\alpha} \in \dot{R}^\times. \tag{1.12}$$

Then, it follows from (1.11), (1.7) and (1.12) that

$$R^0 := \{\alpha \in R \mid (\alpha, \alpha) = 0\} = R \cap \mathbf{Z}^\nu. \tag{1.13}$$

Thus

$$R^\times := R \setminus R^0 = \{\dot{\alpha} + \sigma \mid \dot{\alpha} \in \dot{R}^\times,\ \sigma \in \mathbf{Z}^\nu,\ \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \neq \{0\}\}. \tag{1.14}$$

Note also that from (1.11) and (1.13) we have

$$R^0 = \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}_0 \cap \mathcal{G}^\sigma \neq \{0\}\}. \tag{1.15}$$

The following Lemma is a part of the statement of Proposition 1.20 of [AABGP, III].

Lemma 1.16 *Suppose the triple $(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}})$ satisfies (C1) through (C9). Then the triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ constructed above satisfies axioms (EA1) through (EA4) from Definition I.1.1.*

Proof. First (EA1) follows from (1.4), and (EA2) follows from (1.8) and (1.9).

For (EA3) note that if $\beta \in R$ and $\alpha \in R^\times$, then by (1.14), the α-string through β in R is finite (since $\dot{R}$ is finite). So we have (EA3).

Next by (1.11), R is a discrete subset of $\mathcal{H}^*$ and so we have (EA4). □

Our goal is to make $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ into a (tame) EALA. As we've seen in Lemma 1.16, conditions (C1) through (C9) guarantee that (EA1) through (EA4) holds. We need to put some more conditions on $(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}})$ to force $\mathcal{L}$ to satisfy (EA5). Let X denote the type of the root system $\dot{R}$. Also, as usual, decompose $\dot{R}^\times$ as $\dot{R}^\times = \dot{R}_{sh} \cup \dot{R}_{lg} \cup \dot{R}_{ex}$, where $\dot{R}_{sh}$, $\dot{R}_{lg}$, and $\dot{R}_{ex}$ denote the set of short, long and extra long roots of $\dot{R}^\times$ respectively. As a convention, we assume that

if X is simply laced then any root is a short root.

Therefore in the above decomposition of $\dot{R}^\times$, $\dot{R}_{lg}$ and $\dot{R}_{ex}$ might be empty. Assume that

(C10) if $X \neq C_\ell$, then

$$\mathcal{G}^0 \cap \mathcal{G}_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot{R}_{lg} \text{ and there exists } \tau \in \mathbf{Z}^\nu \text{ such that } \mathcal{G}^\tau \cap \mathcal{G}_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot{R}_{sh},$$

and if $X = C_\ell$, then

$$\mathcal{G}^0 \cap \mathcal{G}_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot{R}_{sh} \text{ and there exists } \tau \in \mathbf{Z}^\nu \text{ such that } \mathcal{G}^\tau \cap \mathcal{G}_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot{R}_{lg}.$$

Note that if $\tau = 0$, then (C10) is equivalent to

$$\mathcal{G}^0 \cap \mathcal{G}_{\dot\alpha} \neq \{0\} \text{ for each } \dot\alpha \in \dot{R}^\times \text{ such that } \frac{1}{2}\dot\alpha \notin \dot{R}. \tag{1.17}$$

(C11) If $\sigma \in R^0$, then there exist $\dot\alpha \in \dot{R}^\times$ and $\eta \in \mathbf{Z}^\nu$ such that $\mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\eta \neq \{0\}$ and $\mathcal{G}_{\dot\alpha} \cap \mathcal{G}^{\eta+\sigma} \neq \{0\}$.

As we will see later these two additional conditions guarantee that (EA5) holds.

Corresponding to the element $\tau \in \mathbf{Z}^\nu$ which appeared in (C10), we define a finite root system $\dot{R}_\tau$ as follows. Let $\epsilon_1, \ldots, \epsilon_{l+1}$ be the usual orthogonal basis for $\mathbf{R}^{l+1}$. Then we can assume that $\dot{R}^{\times}$ has the form

$$
\begin{array}{ll}
\{\pm\epsilon_1\} & \text{if } X = A_1 \\
\{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le \ell+1\} & \text{if } X = A_\ell (l \ge 2) \\
\{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \ne j \le \ell\} & \text{if } X = D_\ell (l \ge 4) \\
\{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \ne j \le \ell\} & \text{if } X = B_\ell (l \ge 2) \\
\{\pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \le i \ne j \le \ell\} & \text{if } X = C_\ell (l \ge 2) \\
\{\pm\epsilon_1, \pm 2\epsilon_1\} & \text{if } X = BC_1 \\
\{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \le i \ne j \le \ell\} & \text{if } X = BC_\ell (l \ge 2) \\
\{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \ne j \le 4\} \cup \{\pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\} & \text{if } X = F_4 \\
\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_3, 2\epsilon_1 - \epsilon_2 - \epsilon_3, 2\epsilon_2 - \epsilon_1 - \epsilon_3, 2\epsilon_3 - \epsilon_1 - \epsilon_2\} & \text{if } X = G_2 \\
\{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \ne j \le 8\} & \\
\quad \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{k(i)}\epsilon_i \mid k(i) = 0, 1, \sum_{i=1}^{8} k(i) \in 2\mathbb{Z}\} & \text{if } X = E_8.
\end{array}
$$

Note that if $X = E_6$ or $X = E_7$, then $\dot{R}^{\times}$ can be identified canonically with a subsystem of E_8.

Definition 1.18 *Let $\dot{R}$ have one of the forms as above and let $\tau \in \mathbb{Z}^{\nu}$. Let $\dot{R}_\tau$ be the set obtained from $\dot{R}$ by changing each ϵ_i to $\epsilon_i + \tau$ if $X \ne C_\ell$ and by changing each ϵ_i to $\epsilon_i + (\tau/2)$ if $X = C_\ell$. Since τ is in the radical of the form, it is clear that $\dot{R}_\tau$ is a finite root system in $Span_{\mathbb{R}}\dot{R}_\tau$ isomorphic to $\dot{R}$. We call $\dot{R}_\tau$, the τ-translation of $\dot{R}$. In fact $\dot{R}_\tau$ has the form*

$$
\begin{array}{ll}
\{\pm(\epsilon_1 + \tau)\} & \textit{if } X = A_1 \\
\{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le \ell+1\} & \textit{if } X = A_\ell (l \ge 2) \\
\{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau) \mid 1 \le i \ne j \le \ell\} & \textit{if } X = D_\ell (l \ge 4) \\
\{\pm(\epsilon_i + \tau), \pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau) \mid 1 \le i \ne j \le \ell\} & \textit{if } X = B_\ell (l \ge 2) \\
\{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + \tau), \pm(2\epsilon_i + \tau) \mid 1 \le i \ne j \le \ell\} & \textit{if } X = C_\ell (l \ge 2) \\
\{\pm(\epsilon_1 + \tau), \pm 2(\epsilon_1 + \tau)\} & \textit{if } X = BC_1 \\
\{\pm(\epsilon_i + \tau), \pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau), \pm 2(\epsilon_i + \tau) \mid 1 \le i \ne j \le \ell\} & \textit{if } X = BC_\ell (l \ge 2) \\
\{\pm(\epsilon_i + \tau), \pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau) \mid 1 \le i \ne j \le 4\} & \\
\qquad \cup \{\pm\frac{1}{2}((\epsilon_1 + \tau) \pm (\epsilon_2 + \tau) \pm (\epsilon_3 + \tau) \pm (\epsilon_4 + \tau))\} & \textit{if } X = F_4 \\
\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_3, 2\epsilon_1 - \epsilon_2 - \epsilon_3, 2\epsilon_2 - \epsilon_1 - \epsilon_3, 2\epsilon_3 - \epsilon_1 - \epsilon_2\} & \textit{if } X = G_2 \\
\{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau) \mid 1 \le i \ne j \le 8\} & \\
\qquad \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{k(i)}(\epsilon_i + \tau) \mid k(i) = 0, 1, \sum_{i=1}^{8} k(i) \in 2\mathbb{Z}\} & \textit{if } X = E_8.
\end{array}
$$

Lastly, we impose the following condition on $(\mathcal{G},(\cdot,\cdot),\dot{\mathcal{H}})$ to make $\mathcal{L}$ into a tame EALA.

(C12) $\mathcal{G}$ is generated as a Lie algebra by $\mathcal{G}_{\dot\alpha}$, $\dot\alpha \in \dot{R}^\times$, or equivalently

$$\mathcal{G}_0 = \sum_{\dot\alpha\in\dot{R}^\times} [\mathcal{G}_{\dot\alpha},\mathcal{G}_{-\dot\alpha}].$$

Remark 1.19 *Condition (C11) is a consequence of conditions (C1)-(C9) and (C12). To see this, first note that by Lemma 1.16, $\mathcal{L}$ satisfies (EA1) through (EA4). Therefore by Theorem I.1.11, $R = -R$. Now let $\sigma \in R^0$. By (1.15), $\mathcal{G}^\sigma \cap \mathcal{G}_0 \neq \{0\}$. By (C12) we have*

$$\mathcal{G}_0 = \sum_{\dot\alpha\in\dot{R}^\times} [\mathcal{G}_{\dot\alpha},\mathcal{G}_{-\dot\alpha}].$$

Thus it follows from (C5) and (C6) that there exists $\zeta,\zeta' \in \mathbb{Z}^\nu$ and $\dot\alpha \in \dot{R}^\times$ such that $\mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\zeta \neq \{0\}$, $\mathcal{G}_{-\dot\alpha} \cap \mathcal{G}^{\zeta'} \neq \{0\}$ and $\zeta + \zeta' = \sigma$. By (1.14), $-\dot\alpha + \zeta' \in R$ and so $\dot\alpha - \zeta' \in R$ (since $R = -R$). Thus $\mathcal{G}_{\dot\alpha} \cap \mathcal{G}^{-\zeta'} \neq \{0\}$, by (1.14) again. Put $\eta = -\zeta'$. Then we have

$$\mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\eta \neq \{0\}, \quad \mathcal{G}_{\dot\alpha} \cap \mathcal{G}^{\sigma+\eta} \neq \{0\}.$$

Hence (C11) holds.

Definition 1.20 *Let $\mathcal{G}$ and $\dot{\mathcal{H}}$ be two complex Lie algebras and $(\cdot,\cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ be a symmetric bilinear form. We call the triple $(\mathcal{G},(\cdot,\cdot),\dot{\mathcal{H}})$ a generalized loop algebra if it satisfies conditions (C1) through (C11) above. We define the core of $\mathcal{G}$, denoted by $\mathcal{G}_c$, to be the subalgebra of $\mathcal{G}$ generated by spaces $\mathcal{G}_{\dot\alpha}$, $\dot\alpha \in \dot{R}^\times$. Indeed,*

$$\mathcal{G}_c = \sum_{\dot\alpha\in\dot{R}^\times} [\mathcal{G}_{\dot\alpha},\mathcal{G}_{-\dot\alpha}] \oplus \sum_{\dot\alpha\in\dot{R}^\times} \mathcal{G}_{\dot\alpha}.$$

If moreover $(\mathcal{G},(\cdot,\cdot),\dot{\mathcal{H}})$ satisfies (C12), we call it a tame generalized loop algebra. When there is no confusion we simply call $\mathcal{G}$ a (tame) generalized loop algebra. If $\mathcal{G}$ is tame, then by (C12), $\mathcal{G}_c = \mathcal{G}$. We define the type of $\mathcal{G}$ to be the type of finite root system $\dot{R}$, defined by (1.1).

The following Proposition is a modified version of Proposition 1.20 of [AABGP, III].

Proposition 1.21 *Let $(\mathcal{G},(\cdot,\cdot),\dot{\mathcal{H}})$ be a generalized loop algebra and $(\mathcal{L},(\cdot,\cdot),\mathcal{H})$ be the triple constructed above. Then $\mathcal{L}$ is an EALA of type X and nullity ν which is tame if*

$\mathcal{G}$ is tame. Moreover, if $\tau = 0$ in (C10), then the root system R of $\mathcal{L}$ is isomorphic to $R(X,S)$, $R(X,S,L)$, $R(X,S,E)$ or $R(X,S,L,E)$ (see Construction I.1.24) according to whether X is simply laced, reduced nonsimply laced, BC_1 or $BC_l (l \geq 2)$ respectively, where

$$\begin{aligned} S &= \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\sigma \neq \{0\}\} \text{ for } \dot\alpha \in \dot R_{sh}, \\ L &= \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\sigma \neq \{0\}\} \text{ for } \dot\alpha \in \dot R_{lg} \ (\text{if } \dot R_{lg} \neq \emptyset), \quad \text{and} \\ E &= \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}_{\dot\alpha} \cap \mathcal{G}^\sigma \neq \{0\}\} \text{ for } \dot\alpha \in \dot R_{ex} \ (\text{if } \dot R_{ex} \neq \emptyset). \end{aligned} \tag{1.22}$$

Proof. Let $\mathcal{G}$ be a generalized loop algebra. By Lemma 1.16 the axioms (EA1) through (EA4) are satisfied. Therefore to show that $\mathcal{L}$ is an EALA, it only remains to prove (EA5). We must show that

(a) $R^\times$ cannot be decomposed as a disjoint union $R_1 \uplus R_2$, where R_1 and R_2 are nonempty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$.

(b) For any $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

We start with (a). Since (EA1) through (EA3) holds, we have from Theorem I.1.11(c) that

$$\begin{gathered} \text{if } \alpha \in R^\times, \text{ then for any } \beta \in R \text{ there exists two non-negative integers} \\ u, d \text{ such that for any } n \in \mathbf{Z}, \text{ we have} \\ \beta + n\alpha \in R \Longleftrightarrow -d \leq n \leq u, \quad \text{and} \quad d - u = \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)}. \end{gathered} \tag{1.23}$$

Now let $\mathcal{V}$ be the real span of R. Then by (1.11), (1.7) and (1.12), the form $(\cdot,\cdot)$ restricts to a real valued form on $\mathcal{V}$. Let $\mathcal{V}^0$ be the kernel of $(\cdot,\cdot)$ restricted to $\mathcal{V}$, let $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and let $^- : \mathcal{V} \longrightarrow \bar{\mathcal{V}}$ be the canonical map. Let $\tau \in \mathbf{Z}^\nu$ be as in (C10) and let $\dot R_\tau$ be the $\tau-$translation of $\dot R$. We claim that

$$\text{if } \dot\alpha \text{ is reduced in } \dot R_\tau, \text{ then } \dot\alpha \in R. \tag{1.24}$$

We first note that by (1.23)

$$R = -R. \tag{1.25}$$

By (C10), (1.14) and (1.25)

$$\begin{aligned} (\dot R_{sh} \pm \tau) \cup \dot R_{lg} \subseteq R \quad & \text{if } X \neq C_\ell \quad \text{and} \\ \dot R_{sh} \cup (\dot R_{lg} \pm \tau) \subseteq R \quad & \text{if } X = C_\ell. \end{aligned} \tag{1.26}$$

This in particular shows that

$$\begin{gathered}\text{if } \dot\alpha \in \dot R^\times \text{ is reduced then there exists } n \in \{0, \pm 1\} \text{ so that } \pm\dot\alpha \pm n\tau \in R. \\ \text{This in turn gives } n\tau \in (\mathrm{Span}_{\mathbb{R}} R \cap \mathbb{Z}^\nu) \subseteq \mathrm{Span}_{\mathbb{R}} R^0.\end{gathered} \tag{1.27}$$

Now if $X = A_1$ or $X = BC_1$, (1.24) is satisfied, by (1.26). If $X = B_\ell (l \geq 3)$ or $X = BC_\ell (l \geq 2)$, then the set of reduced roots in $\dot R_\tau$ is

$$\{\pm(\epsilon_i + \tau), \pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + 2\tau) \mid 1 \leq i \neq j \leq \ell\}.$$

By (1.26) we only need to show that $\pm(\epsilon_i + \epsilon_j + 2\tau) \in R$ for $1 \leq i \neq j \leq \ell$. Fix $i \neq j$. We have $\epsilon_i - \epsilon_j \in R$ and $\epsilon_i + \tau \in R$. It can be easily seen that $\epsilon_i + \epsilon_j + 2\tau$ is in the $(\epsilon_j + \tau)$–string through $\epsilon_i - \epsilon_j$. Therefore, by (1.23), $\epsilon_i + \epsilon_j + 2\tau \in R$. Thus 1.24 holds in this case.

Next let $X = C_\ell (l \geq 2)$. We have

$$\dot R_\tau^\times = \{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j + \tau), \pm(2\epsilon_i + \tau) \mid 1 \leq i \neq j \leq \ell\}.$$

By (1.26), we only need to show that $\pm(\epsilon_i + \epsilon_j + \tau) \in R$ for $1 \leq i \neq j \leq \ell$. Fix $i \neq j$. We have $\epsilon_i - \epsilon_j \in R$ and $2\epsilon_j + \tau \in R$. It can be easily seen that $\epsilon_i + \epsilon_j + \tau$ is in the $(2\epsilon_j + \tau)$–string through $\epsilon_i - \epsilon_j$. By (1.23), $\epsilon_i + \epsilon_j + \tau \in R$. Thus (1.24) holds in this case too.

For the remaining types we can use similar arguments to see that (1.24) holds in these cases too. This finishes the proof of (1.24).

Next let $\dot{\mathcal{V}}_\tau$ be the real span of $\dot R_\tau$. By (1.24), $\dot{\mathcal{V}}_\tau \subseteq \mathcal{V}$. Hence $\dot{\mathcal{V}}_\tau + \mathcal{V}^0 \subseteq \mathcal{V}$. We now show the reverse inclusion. We need only to show that $R \subseteq \dot{\mathcal{V}}_\tau + \mathcal{V}^0$. Let $\alpha \in R$. By (1.14) and (1.15), $\alpha = \dot\alpha + \sigma$ for some $\dot\alpha \in \dot R$ and $\sigma \in \mathbb{Z}^\nu$. By definition of $\dot R_\tau$ there exists $n \in \{0, \pm 1, \pm 2\}$ such that $\dot\alpha + n\tau \in \dot R_\tau$. Now since

$$\dot\alpha + \sigma = (\dot\alpha + n\tau) + (\sigma - n\tau) \in R \subseteq \mathcal{V} \quad \text{and} \quad \dot\alpha + n\tau \in \dot R_\tau \subseteq \dot{\mathcal{V}}_\tau \subseteq \mathcal{V},$$

we get $\sigma - n\tau \in \mathcal{V} \cap \mathbb{Z}^\nu$. Thus $\sigma - n\tau \in \mathcal{V}^0$. This gives $\alpha = \dot\alpha + \sigma \in \dot{\mathcal{V}}_\tau + \mathcal{V}^0$. Hence $\mathcal{V} = \dot{\mathcal{V}}_\tau + \mathcal{V}^0$. But this sum is direct because of (C4) and the fact that $\dot R_\tau \cong \dot R$. Thus $\mathcal{V} = \dot{\mathcal{V}}_\tau \oplus \mathcal{V}^0$ and so the restriction of $^-$ to $\dot{\mathcal{V}}_\tau$ is an isometry. Moreover the argument we used to prove that $\mathcal{V} \subseteq \dot{\mathcal{V}}_\tau + \mathcal{V}^0$ shows that this map sends $\dot R_\tau$ to $\bar R$, the image of R under

$^{-}$. Hence $\bar{R}$ is an irreducible finite root system in $\bar{\mathcal{V}}$ (which is isomorphic to $\dot{R}$). Now (a) follows easily from this fact.

To show (b), let $\sigma \in R^0$. Then by (C11), there exist $\dot{\alpha} \in \dot{R}^\times$ and $\eta \in \mathbf{Z}^\nu$ such that $\mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\eta \neq \{0\}$ and $\mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^{\sigma+\eta} \neq \{0\}$. Then by (1.14), $\alpha := \dot{\alpha}+\eta \in R$ and $\alpha+\sigma = \dot{\alpha}+\eta+\sigma \in R$. Therefore (b) holds.

Thus, $\mathcal{L}$ is an EALA. Since we have seen $\bar{R} \cong \dot{R}$, it follows that R (and hence by definition $\mathcal{L}$) has type X.

Next, we show that R has nullity ν. For this we need to check that $\mathcal{V}^0$ is ν-dimensional. By (1.11) and (1.13), $R^0 \subseteq \mathcal{V}^0 \subseteq \mathbf{R}^\nu$. Therefore by (C8), we will have $\mathcal{V}^0$ is ν-dimensional if we prove that

$$\{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}^\sigma \neq \{0\}\} \subseteq \mathrm{Span}_{\mathbf{R}} R^0. \tag{1.28}$$

By (C5), (C6) and (1.15)

$$\{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}^\sigma \neq \{0\}\} \subseteq R^0 \cup (\cup_{\dot{\alpha} \in \dot{R}^\times} \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \neq \{0\}\}).$$

Therefore, (1.28) follows if we prove that

$$\mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \neq \{0\},\ \sigma \in \mathbf{Z}^\nu,\ \dot{\alpha} \in \dot{R}^\times \text{ implies } \sigma \in \mathrm{Span}_{\mathbf{R}} R^0. \tag{1.29}$$

So let $\sigma \in \mathbf{Z}^\nu$ and $\mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \neq \{0\}$ for some $\dot{\alpha} \in \dot{R}^\times$. By (1.14), $\dot{\alpha} + \sigma \in R$. If $\dot{\alpha}$ is reduced in $\dot{R}$, then by (1.27), there exists $n \in \mathbf{Z}$ such that $\pm\dot{\alpha} + n\tau \in R$. Now consider the $(\pm\dot{\alpha} + n\tau)$-string through $\dot{\alpha} + \sigma$ to get that $\sigma \pm n\tau \in R$. Thus $\sigma \pm n\tau \in R \cap \mathbf{Z}^\nu = R^0$. Hence $\sigma \in \mathrm{Span}_{\mathbf{R}} R^0$. If $\dot{\alpha}$ is not reduced in $\dot{R}$ then $\dot{\alpha} = 2\dot{\beta}$ for some reduced $\dot{\beta}$ in $\dot{R}$. Then by (1.27), $\pm\dot{\beta} + n\tau \in R$ for some $n \in \mathbf{Z}$. Now consider the $(\pm\dot{\beta} + n\tau)$-string through $\dot{\alpha} + \sigma$ to get that $\sigma \pm 2n\tau \in R$. Thus $\sigma \pm 2n\tau \in R \cap \mathbf{Z}^\nu = R^0$. This gives $\sigma \in \mathrm{Span}_{\mathbf{R}} R^0$ and finishes the proof of (1.29). Hence R has nullity ν.

Next, we want to show that if $\mathcal{G}$ is tame , so is $\mathcal{L}$. First, we make the following claim.

Claim. If $\mathcal{L}_c$ and $\mathcal{G}_c$ are the cores of $\mathcal{L}$ and $\mathcal{G}$ respectively, then $\mathcal{L}_c = \mathcal{G}_c \oplus \mathcal{C}$.

Proof of claim. By (C6), (1.10) and (1.14), we have that $\mathcal{L}_c$ is the subalgebra of $\mathcal{L}$ generated by $\mathcal{G}_{\dot{\alpha}}, \dot{\alpha} \in \dot{R}^\times$. Thus from the way the bracket is defined on $\mathcal{L}$, we get $\mathcal{L}_c \subseteq \mathcal{G}_c \oplus \mathcal{C}$. For the reverse inclusion note first that By (C6) and (1.10),

$$\mathcal{G}_{\dot{\alpha}} = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}_{\dot{\alpha}+\sigma} \subseteq \sum_{\alpha \in R^\times} \mathcal{L}_\alpha \subseteq \mathcal{L}_c, \quad \text{for all } \dot{\alpha} \in \dot{R}^\times.$$

Therefore $\mathcal{G}_c \subseteq \mathcal{L}_c$. Thus it only remains to show that $\mathcal{C} \subseteq \mathcal{L}_c$. Let $\sigma \in R^0$. By (I.1.32), $t_\sigma \in \mathcal{L}_c$. By (C8) and (1.28), R^0 contains a basis of $\mathbf{R}^\nu$. Thus $t_\sigma \in \mathcal{L}_c$ for all $\sigma \in \mathbf{Z}^\nu$ and so $c_i = t_{\delta_i} \in \mathcal{L}_c$ for $i = 1, \ldots, \nu$. Hence $\mathcal{C} \subseteq \mathcal{L}_c$ and so

$$\mathcal{L}_c = \mathcal{G}_c \oplus \mathcal{C}$$

as claimed.

Now if $\mathcal{G}$ is tame, then (C12) holds and so $\mathcal{G}_c = \mathcal{G}$. Thus $\mathcal{L}_c = \mathcal{G} \oplus \mathcal{C}$, by the above claim. Therefore the orthogonal complement $\mathcal{L}_c^\perp$ of $\mathcal{L}_c$ in $\mathcal{L}$ (see (I.1.41) for definition) is $\mathcal{C}$, which is contained in $\mathcal{L}_c$. So $\mathcal{L}$ is tame.

Finally if the element τ appearing in (C10) is zero, then the last statement of the proposition follows from Theorem 2.37 of [AABGP, II]. □

We now would like to state some results regarding the core $\mathcal{G}_c$ of a generalized loop algebra $\mathcal{G}$. So in what follows let $(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}})$ be a generalized loop algebra and $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ be the EALA constructed above.

From the claim stated in the proof of Proposition 1.21, we get the following corollary.

Corollary 1.30 $\mathcal{L}_c = \mathcal{G}_c \oplus \mathcal{C}$. □

Since for $\dot\alpha \in \dot R^\times$, we have $[\mathcal{G}_{\dot\alpha}, \mathcal{G}_0] \subseteq \mathcal{G}_{\dot\alpha} \subseteq \mathcal{G}_c$, it follows that

$$\mathcal{G}_c \text{ is an ideal of } \mathcal{G}.$$

Corollary 1.31 *The core $\mathcal{G}_c$ of $\mathcal{G}$ is perfect. In particular any tame generalized loop algebra is perfect.*

Proof. We only need to show that $\mathcal{G}_c \subseteq [\mathcal{G}_c, \mathcal{G}_c]$. By Proposition 1.21, $\mathcal{L} = \mathcal{G} \oplus \mathcal{C} \oplus \mathcal{D}$ is an EALA. By Lemma I.1.33, $\mathcal{L}_c$, the core of $\mathcal{L}$, is perfect. Thus

$$\mathcal{G}_c \oplus \mathcal{C} = \mathcal{L}_c = [\mathcal{L}_c, \mathcal{L}_c]' = [\mathcal{G}_c \oplus \mathcal{C}, \mathcal{G}_c \oplus \mathcal{C}]' \subseteq [\mathcal{G}_c, \mathcal{G}_c] \oplus \mathcal{C}.$$

Hence $\mathcal{G}_c \subseteq [\mathcal{G}_c, \mathcal{G}_c]$. □

Let

$$\mathcal{G}_c^\perp = \{x \in \mathcal{G} \mid (x, \mathcal{G}_c) = \{0\}\}.$$

Since $\mathcal{G}_c$ is an ideal of $\mathcal{G}$ which is perfect, it follows easily from invariancy and nondegeneracy of the form that

$$\mathcal{G}_c^{\perp} = \{x \in \mathcal{G} \mid [x, \mathcal{G}_c] = \{0\}\}. \tag{1.32}$$

Corollary 1.33 $\mathcal{G}_c^{\perp} \oplus \mathcal{C} \subseteq \mathcal{L}_c^{\perp}$.

Proof. We have $\mathcal{L}_c = \mathcal{G}_c \oplus \mathcal{C}$. Thus

$$(\mathcal{L}_c, \mathcal{G}_c^{\perp} \oplus \mathcal{C}) = (\mathcal{G}_c \oplus \mathcal{C}, \mathcal{G}_c^{\perp} \oplus \mathcal{C}) = (\mathcal{G}_c, \mathcal{G}_c^{\perp}) = \{0\}.$$

□

Lemma 1.34 *Let $(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}})$ be a generalized loop algebra such that $\dot{\mathcal{H}} \subseteq \mathcal{G}_c$ and the form $(\cdot,\cdot)$ restricted to $\mathcal{G}_c \times \mathcal{G}_c$ is nondegenerate. Then $(\mathcal{G}_c, (\cdot,\cdot), \dot{\mathcal{H}})$ is a tame generalized loop algebra.*

Proof. We have (C1)-(C11) hold for $\mathcal{G}$ and we want to show that (C1)- (C12) hold for $\mathcal{G}_c$. Clearly (C1), (C2), (C3) and (C4) hold for $\mathcal{G}_c$. Moreover, we have

$$\mathcal{G}_c = \sum_{\dot\alpha \in \dot R^{\times}} (\mathcal{G}_c)_{\dot\alpha} \quad \text{where } (\mathcal{G}_c)_{\dot\alpha} = \{x \in \mathcal{G}_c \mid [h, x] = \dot\alpha(h)x \text{ for all } h \in \dot{\mathcal{H}}\}.$$

Indeed,

$$(\mathcal{G}_c)_{\dot\alpha} = \mathcal{G}_{\dot\alpha} \text{ if } \dot\alpha \in \dot R^{\times} \quad \text{and} \quad (\mathcal{G}_c)_0 = \mathcal{G}_c \cap \mathcal{G}_0 = \sum_{\dot\alpha \in \dot R^{\times}} [\mathcal{G}_{\dot\alpha}, \mathcal{G}_{-\dot\alpha}].$$

Since $\mathcal{G} = \sum_{\sigma \in \mathbf{Z}^{\nu}} \mathcal{G}^{\sigma}$ and by (C6), $\mathcal{G}_c$ is generated by homogenous elements with respect to the $\mathbf{Z}^{\nu}$-grading on $\mathcal{G}$, we have

$$\mathcal{G}_c = \oplus \sum_{\sigma \in \mathbf{Z}^{\nu}} \mathcal{G}_c^{\sigma} \quad \text{where} \quad \mathcal{G}_c^{\sigma} = \mathcal{G}_c \cap \mathcal{G}^{\sigma}, \ \ \sigma \in \mathbf{Z}^{\nu}.$$

Thus (C5) holds for $\mathcal{G}_c$. Since (C7) holds for $\mathcal{G}$, we have $\dot{\mathcal{H}} \subseteq \mathcal{G}^0$. Thus $\dot{\mathcal{H}} \subseteq \mathcal{G}^0 \cap \mathcal{G}_c = (\mathcal{G}_c)^0$. Hence (C6) holds for $\mathcal{G}_c$ by Lemma I.1.56. Next, we have $\dot{\mathcal{H}} = \mathcal{G}^0 \cap \mathcal{G}_0$ and $\dot{\mathcal{H}} \subseteq \mathcal{G}_c$, so

$$\dot{\mathcal{H}} = \mathcal{G}^0 \cap \mathcal{G}_0 = (\mathcal{G}^0 \cap \mathcal{G}_c) \cap (\mathcal{G}_0 \cap \mathcal{G}_c) = (\mathcal{G}_c)^0 \cap (\mathcal{G}_c)_0.$$

Thus (C7) holds for $\mathcal{G}_c$. Next, we show (C8) hols for $\mathcal{G}_c$. Since (C8) holds for $\mathcal{G}$, we have $\{\sigma \in \mathbf{Z}^{\nu} \mid \mathcal{G}^{\sigma} \neq \{0\}\}$ generates a subgroup of rank ν of $\mathbf{Z}^{\nu}$. So it is enough to show that the set $\{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}^\sigma \neq \{0\}\ \}$ is contained in the subgroup of $\mathbb{Z}^\nu$ generated by the set $\{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}_c^\sigma \neq \{0\}\ \}$. Let $\xi \in \mathbb{Z}^\nu$ and $\mathcal{G}^\xi \neq \{0\}$. Then $\{0\} \neq \mathcal{G}^\xi \subseteq \mathcal{G} = \sum_{\dot\alpha\in\dot R}\sum_{\sigma\in\mathbb{Z}^\nu}\mathcal{G}_{\dot\alpha}\cap\mathcal{G}^\sigma$. Thus $\mathcal{G}^\xi \cap \mathcal{G}_{\dot\alpha} \neq \{0\}$ for some $\dot\alpha \in \dot R$. If $\dot\alpha \in \dot R^\times$, then

$$\{0\} \neq \mathcal{G}^\xi \cap \mathcal{G}_{\dot\alpha} \subseteq \mathcal{G}^\xi \cap \mathcal{G}_c = \mathcal{G}_c^\xi$$

and so

$$\xi \in \{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}_c^\sigma \neq \{0\}\}.$$

If $\dot\alpha = 0$, then $\mathcal{G}^\xi \cap \mathcal{G}_0 \neq \{0\}$. Since (C11) holds for $\mathcal{G}$, there exist $\dot\beta \in \dot R^\times$ and $\eta \in \mathbb{Z}^\nu$ such that $\mathcal{G}_{\dot\beta} \cap \mathcal{G}^\eta \neq \{0\}$ and $\mathcal{G}_{\dot\beta} \cap \mathcal{G}^{\xi+\eta} \neq \{0\}$. Thus

$$\eta, \eta + \xi \in \{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}_c^\sigma \neq \{0\}\}.$$

Hence ξ belongs to the subgroup of $\mathbb{Z}^\nu$ generated by $\{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}_c^\sigma \neq \{0\}\}$. Hence (C8) holds for $\mathcal{G}_c$. (C9) holds for $\mathcal{G}_c$ since it holds for $\mathcal{G}$. Since $(\mathcal{G}_c)_{\dot\alpha} = \mathcal{G}_{\dot\alpha}$ for $\dot\alpha \in \dot R^\times$, (C10) holds for $\mathcal{G}_c$. (C12) holds for $\mathcal{G}_c$ since

$$\mathcal{G}_c = \sum_{\dot\alpha\in\dot R^\times}[\mathcal{G}_{\dot\alpha}, \mathcal{G}_{-\dot\alpha}] \oplus \sum_{\dot\alpha\in\dot R^\times}\mathcal{G}_{\dot\alpha} = \sum_{\dot\alpha\in\dot R^\times}[(\mathcal{G}_c)_{\dot\alpha}, (\mathcal{G}_c)_{-\dot\alpha}] \oplus \sum_{\dot\alpha\in\dot R^\times}(\mathcal{G}_c)_{\dot\alpha}.$$

By Remark 1.19, (C11) also holds. □

We now state a converse result to Proposition 1.21. Let us denote by $\mathcal{Z}(\mathcal{Y})$, the center of a Lie algebra $\mathcal{Y}$. Recall the definition of a nondegenerate EALA from Definition I.1.38.

Proposition 1.35 *Let $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ be a nondegenerate tame extended affine Lie algebra. Let $\mathcal{L}_c$ be the core of $\mathcal{L}$ and set*

$$\mathcal{G}_c := \frac{\mathcal{L}_c}{\mathcal{Z}(\mathcal{L}_c)}, \qquad \dot{\mathcal{H}}_c := \frac{(\mathcal{H}\cap\mathcal{L}_c) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} \quad \textit{and let}$$

$(\cdot,\cdot)_c$ be the induced bilinear form from $\mathcal{L}$ on $\mathcal{G}_c$.

Then the triple $(\mathcal{G}_c, (\cdot,\cdot)_c, \dot{\mathcal{H}}_c)$ is a tame generalized loop algebra.

Proof. Let $^- : \mathcal{L}_c \to \frac{\mathcal{L}_c}{\mathcal{Z}(\mathcal{L}_c)}$ be the canonical quotient map. Let us denote by $\bar x$ the image of an element $x \in \mathcal{L}_c$ under the map $^-$. Define

$$[\bar x, \bar y] := \overline{[x, y]} = [x, y] + \mathcal{Z}(\mathcal{L}_c) \text{ for all } x, y \in \mathcal{L}_c. \tag{1.36}$$

Since $\mathcal{L}$ is tame, $\mathcal{Z}(\mathcal{L}_c) = \mathcal{L}_c^{\perp}$ and so $(\cdot,\cdot)_c$ is a well-defined symmetric bilinear form on $\mathcal{G}_c \times \mathcal{G}_c$. We must show that the triple $(\mathcal{G}_c, (\cdot,\cdot)_c, \dot{\mathcal{H}}_c)$ satisfies (C1)-(C12).

Since the form $(\cdot,\cdot)$ is symmetric and invariant, so is $(\cdot,\cdot)_c$. Since $\mathcal{Z}(\mathcal{L}_c) = \mathcal{L}_c^{\perp}$, $(\cdot,\cdot)_c$ is nondegenerate. So (C1) holds. Clearly $\dot{\mathcal{H}}_c$ is a finite dimensional abelian subalgebra of $\mathcal{G}_c$. We show that $\mathrm{ad}_{\mathcal{G}_c}\dot{\mathcal{H}}_c$ acts diagonally on $\mathcal{G}_c$. Let R be the root system of $\mathcal{L}$ as in Chapter I. From (I.1.31), we have $\mathcal{L}_c = \sum_{\alpha\in R}\mathcal{L}_\alpha \cap \mathcal{L}_c$. Thus $\{\bar{x} \mid x \in \mathcal{L}_\alpha \cap \mathcal{L}_c\}$ spans $\mathcal{G}_c$ and so cantains a basis of $\mathcal{G}_c$. Now for $x \in \mathcal{L}_\alpha \cap \mathcal{L}_c$, $\alpha \in R$, and $h \in \mathcal{H} \cap \mathcal{L}_c$ we have $[\bar{h}, \bar{x}] = \overline{[h,x]} = \alpha(h)\bar{x}$. Thus $\mathrm{ad}_{\dot{\mathcal{G}}_c}\dot{\mathcal{H}}_c$ acts diagonally and so (C2) holds. Recall the vector spaces $\dot{\mathcal{V}}_{\mathbf{C}}$ and $\mathcal{V}_{\mathbf{C}}^0$ from (I.1.35). From (I.1.37) we have $(\cdot,\cdot)$ is nondegenerate on $\dot{\mathcal{V}}_{\mathbf{C}}$ and from (I.1.44) we have $\mathcal{V}_{\mathbf{C}}^0 \subseteq \mathcal{Z}(\mathcal{L}_c)$. By Lemma I.1.39,

$$\dot{\mathcal{H}}_c = \frac{(\dot{\mathcal{H}} \cap \mathcal{L}_c) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} = \frac{(\dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}_{\mathbf{C}}^0) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} = \frac{\dot{\mathcal{V}}_{\mathbf{C}} + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)}. \tag{1.37}$$

Thus the restriction of the form $(\cdot,\cdot)_c$ to $\dot{\mathcal{H}}_c$ is nondegenerate. So (C3) also holds. As usual we transfer this form to $\dot{\mathcal{H}}_c^*$. Since (C2) holds we have

$$\begin{gathered} \mathcal{G}_c = \textstyle\sum_{\dot{\alpha}\in\dot{\mathcal{H}}_c^*}(\mathcal{G}_c)_{\dot{\alpha}} \quad \text{where} \\ (\mathcal{G}_c)_{\dot{\alpha}} = \{\bar{x} \in \mathcal{G}_c \mid [\bar{h}, \bar{x}] = \dot{\alpha}(\bar{h})\bar{x}, \text{ for all } \bar{h} \in \dot{\mathcal{H}}_c\}\,. \end{gathered} \tag{1.38}$$

Put

$$\dot{R}_c = \{\dot{\alpha} \in \dot{\mathcal{H}}_c^* \mid (\mathcal{G}_c)_{\dot{\alpha}} \neq \{0\}\}.$$

We have $\{0\} \neq \dot{\mathcal{H}}_c \subseteq (\mathcal{G}_c)_0$, so $0 \in \dot{R}_c$. We also put $\dot{R}_c^{\times} = \dot{R}_c \setminus \{0\}$. To go on we need to introduce some new notation. For $\alpha \in R$, define $\bar{\alpha} \in \dot{\mathcal{H}}_c^*$ by

$$\bar{\alpha}(\bar{h}) = (\bar{t}_\alpha, \bar{h})_c \text{ for all } \bar{h} \in \dot{\mathcal{H}}_c.$$

(Recall the definition of t_α from (I.1.5) and note that $\bar{t_\alpha} \in \dot{\mathcal{H}}_c$ by (I.1.32)). Let

$$\bar{R}_c := \{\bar{\alpha} \mid \alpha \in R\}.$$

Then for $\alpha \in R$, we have

$$\bar{\alpha} = 0 \text{ iff } \alpha \text{ is isotropic}. \tag{1.39}$$

Indeed,

$$\begin{aligned}\bar{\alpha} = 0 &\iff \bar{\alpha}(\bar{h}) = 0 \text{ for all } \bar{h} \in \dot{\mathcal{H}}_c\\ &\iff (\bar{t}_\alpha, \bar{h})_c = 0 \text{ for all } \bar{h} \in \dot{\mathcal{H}}_c\\ &\iff (t_\alpha, h) = 0 \text{ for all } h \in \mathcal{H} \cap \mathcal{L}_c = \dot{\mathcal{V}}_{\mathbf{C}} \oplus \mathcal{V}^0_{\mathbf{C}} \quad \text{(see Lemma I.1.39)}\\ &\iff t_\alpha \in \mathcal{V}^0_c\\ &\iff \alpha \text{ is isotropic.}\end{aligned}$$

Let

$$\bar{\mathcal{L}}_\alpha = \{\bar{x} \mid x \in \mathcal{L}_\alpha \cap \mathcal{L}_c\} = \frac{(\mathcal{L}_\alpha \cap \mathcal{L}_c) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)}, \quad (\alpha \in R).$$

Then

$$\bar{\mathcal{L}}_\alpha \subseteq (\mathcal{G}_c)_{\bar{\alpha}} \text{ for all } \alpha \in R. \tag{1.40}$$

Next, we need to prove that

$$\bar{R}_c = \dot{R}_c \quad \text{and} \quad (\mathcal{G}_c)_{\bar{\alpha}} = \sum_{\substack{\beta \in R\\ \bar{\beta} = \bar{\alpha}}} \bar{\mathcal{L}}_\beta. \tag{1.41}$$

By (1.40), $\bar{\mathcal{L}}_\beta \subseteq (\mathcal{G}_c)_{\bar{\alpha}}$ for any $\beta \in R$ with $\bar{\beta} = \bar{\alpha}$. Thus

$$\sum_{\substack{\beta \in R\\ \bar{\beta} = \bar{\alpha}}} \bar{\mathcal{L}}_\beta \subseteq (\mathcal{G}_c)_{\bar{\alpha}}. \tag{1.42}$$

Since $\mathcal{G}_c = \sum_{\beta \in R} \bar{\mathcal{L}}_\beta$, we have, using (1.42),

$$\sum_{\dot{\alpha} \in \dot{R}_c} (\mathcal{G}_c)_{\dot{\alpha}} = \mathcal{G}_c = \sum_{\beta \in R} \bar{\mathcal{L}}_\beta = \sum_{\alpha \in R} \Big(\sum_{\substack{\beta \in R\\ \bar{\beta} = \bar{\alpha}}} \bar{\mathcal{L}}_\beta \Big) \subseteq \sum_{\alpha \in R} (\mathcal{G}_c)_{\bar{\alpha}} \subseteq \mathcal{G}_c.$$

Thus

$$\sum_{\dot{\alpha} \in \dot{R}_c} (\mathcal{G}_c)_{\dot{\alpha}} = \mathcal{G}_c = \sum_{\alpha \in R} (\mathcal{G}_c)_{\bar{\alpha}}. \tag{1.43}$$

Since $\bar{R}_c \subseteq \dot{R}_c$, (1.43) gives (1.41).

For (C4), note that, using Lemmas I.1.39 and I.1.44, we have $\dot{\mathcal{H}}_c \cong \dot{\mathcal{V}}_{\mathbf{C}}$. Let $\dot{R}$ be the finite root system defined by (I.1.18). Then the assignment $\bar{\alpha} \mapsto \alpha|_{\dot{\mathcal{V}}_{\mathbf{C}}}$ defines a linear isometry from $\dot{\mathcal{H}}_c$ onto $\dot{\mathcal{V}}_{\mathbf{C}}$ which maps $\bar{R}_c = \dot{R}_c$ onto $\dot{R}$. This gives (C4). For (C5), let $\mathcal{L} = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}^\sigma$ be the $\mathbf{Z}^\nu$-grading on $\mathcal{L}$ defined by (I.1.55). By (I.1.60), $\mathcal{L}_c = \sum_{\sigma \in \mathbf{Z}^\nu} \mathcal{L}_c \cap \mathcal{L}^\sigma$ and so

$$\mathcal{G}_c = \frac{\mathcal{L}_c}{\mathcal{Z}(\mathcal{L}_c)} = \sum_{\sigma \in \mathbf{Z}^\nu} \frac{(\mathcal{L}_c \cap \mathcal{L}^\sigma) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)}.$$

Thus $\mathcal{G}_c = \bigoplus \sum_{\sigma\in\mathbb{Z}^\nu} \mathcal{G}_c^\sigma$ where $\mathcal{G}_c^\sigma = \frac{(\mathcal{L}_c\cap\mathcal{L}^\sigma)+\mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)}$. Clearly $\mathcal{G}_c^\sigma \cong \frac{\mathcal{L}_c\cap\mathcal{L}^\sigma}{\mathcal{Z}(\mathcal{L}_c)\cap\mathcal{L}^\sigma}$. For (C6), we must show

$$(\mathcal{G}_c)_{\bar{\alpha}} = \sum_{\sigma\in\mathbb{Z}^\nu} (\mathcal{G}_c)_{\bar{\alpha}} \cap \mathcal{G}_c^\sigma \quad \text{for } \alpha \in R.$$

By Lemma I.1.56, we only need to show that $\dot{\mathcal{H}}_c \subseteq \mathcal{G}_c^0$. We have $\dot{\mathcal{V}}_{\mathbf{C}} \subseteq \mathcal{L}^0$ by (I.1.55). Thus using (1.37), we have

$$\dot{\mathcal{H}}_c = \frac{\dot{\mathcal{V}}_{\mathbf{C}} + \mathbf{Z}(\mathcal{L}_c)}{\mathbf{Z}(\mathcal{L}_c)} \subseteq \frac{(\mathcal{L}_c \cap \mathcal{L}^0) + \mathbf{Z}(\mathcal{L}_c)}{\mathbf{Z}(\mathcal{L}_c)} = \mathcal{G}_c^0.$$

For (C7), we must show $\dot{\mathcal{H}}_c = \mathcal{G}_c^0 \cap (\mathcal{G}_c)_{\bar{0}}$. By (1.39) and (1.41)

$$(\mathcal{G}_c)_{\bar{0}} = \sum_{\substack{\sigma\in R^0\\ \bar{\sigma}=0}} \bar{\mathcal{L}}_\sigma = \sum_{\sigma\in R^0} \bar{\mathcal{L}}_\sigma = \frac{\sum_{\sigma\in R^0}(\mathcal{L}_\sigma \cap \mathcal{L}_c) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)}.$$

and by Lemma I.1.43, $\mathcal{Z}(\mathcal{L}_c) \subseteq \sum_{\sigma\in R^0} \mathcal{L}_\sigma$. Thus we have

$$\begin{aligned}
\mathcal{G}_c^0 \cap (\mathcal{G}_c)_{\bar{0}} &= \frac{(\mathcal{L}_c \cap \mathcal{L}^0) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} \bigcap \frac{\sum_{\sigma\in R^0}(\mathcal{L}_\sigma \cap \mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} \\
&= \frac{(\mathcal{L}_c \cap \mathcal{L}^0 \cap \mathcal{L}_0) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} = \frac{(\mathcal{H} \cap \mathcal{L}_c) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} = \dot{\mathcal{H}}_c.
\end{aligned}$$

This gives (C7). For (C8), let $\dot{\alpha} \in \dot{R}_{sh}$ and $S := S_{\dot{\alpha}} = \{\sigma \in R^0 \mid \dot{\alpha} + \sigma \in R\}$. By (I.1.22), S is a semilattice of rank ν. Let $\sigma \in S$. Then $\dot{\alpha} + \sigma \in R^\times$ and $\mathcal{L}_{\dot{\alpha}+\sigma} \subseteq \mathcal{L}_c \cap \mathcal{L}^\sigma$. So

$$\{0\} \neq \mathcal{L}_{\dot{\alpha}+\sigma} \subseteq \frac{(\mathcal{L}_c \cap \mathcal{L}^\sigma) + \mathcal{Z}(\mathcal{L}_c)}{\mathcal{Z}(\mathcal{L}_c)} = (\mathcal{G}_c)^\sigma.$$

Thus $S \subseteq \{\sigma \in \mathbb{Z}^\nu \mid \mathcal{G}_c^\sigma \neq \{0\}\}$. Thus (C8) holds. We now show (C9) holds. Let $\sigma, \tau \in \mathbb{Z}^\nu$ and $(\mathcal{G}_c^\sigma, \mathcal{G}_c^\tau) \neq \{0\}$. Then $(\mathcal{L}^\sigma, \mathcal{L}^\tau) \neq \{0\}$. So using (I.1.59), there exist $\alpha, \beta \in R$ and $x \in \mathcal{L}^\sigma \cap \mathcal{L}_\alpha$, $y \in \mathcal{L}^\tau \cap \mathcal{L}_\beta$, so that $(x, y) \neq \{0\}$. By (I.1.4), $\alpha + \beta = 0$. If $\alpha = \dot{\alpha} + \sum_{i=1}^\nu m_i\delta_i$, $\dot{\alpha} \in \dot{R}$, then $\beta = -\dot{\alpha} - \sum_{i=1}^\nu m_i\delta_i$. So $\sigma = (m_1, \ldots, m_\nu)$ and $\tau = (-m_1, \ldots, -m_\nu)$. Thus $\sigma + \tau = 0$. This gives (C9).

Next, we show that (C10) holds with $\tau = 0$. Therefore we only need to show that (1.17) holds. Recall from (I.1.18), the finite root system $\dot{R}$ attached to the root system R of $\mathcal{L}$. Let $\alpha \in R$, $\bar{\alpha} \in \bar{R}_c \setminus \{0\}$ and $\frac{1}{2}\bar{\alpha} \notin \bar{R}_c$. By (1.39), $\alpha \in R^\times$. Thus $\alpha = \dot{\alpha} + \sigma$ for some $\dot{\alpha} \in \dot{R}^\times$ and $\sigma \in R^0$. If $\dot{\alpha}$ is not reduced in $\dot{R}$, then $\frac{1}{2}\dot{\alpha}$ is reduced. By (I.1.20), $\frac{1}{2}\dot{\alpha} \in R$. Thus $\frac{1}{2}\bar{\dot{\alpha}} \in \bar{R}_c$. But $\frac{1}{2}\bar{\alpha} = \frac{1}{2}\bar{\dot{\alpha}} \in \bar{R}_c$ which is a contradiction. Thus $\dot{\alpha}$ is reduced and so $\dot{\alpha} \in R$.

by (I.1.20). We have $\mathcal{L}_{\dot{\alpha}} \cap \mathcal{Z}(\mathcal{L}_c) = \{0\}$, by Lemma I.1.43 and so $\bar{\mathcal{L}_{\dot{\alpha}}} \neq \{0\}$. By (I.1.55), $\mathcal{L}_{\dot{\alpha}} \subseteq \mathcal{L}^0 \cap \mathcal{L}_c$ and so $\bar{\mathcal{L}_{\dot{\alpha}}} \subseteq \mathcal{G}_c^0$. Also $\bar{\mathcal{L}}_{\dot{\alpha}} \subseteq \mathcal{L}_{\bar{\dot{\alpha}}} = \mathcal{L}_{\bar{\alpha}} \subseteq (\mathcal{G}_c)_{\bar{\alpha}}$. Hence

$$\{0\} \neq \bar{\mathcal{L}}_{\dot{\alpha}} \subseteq (\mathcal{G}_c)^0 \cap (\mathcal{G}_c)_{\bar{\alpha}},$$

which gives (C10).

By Remark 1.19, (C11) is automatically satisfied if we show that (C12) holds. Since $\mathcal{G}_c = \sum_{\bar{\alpha} \in \bar{R}_c} (\mathcal{G}_c)_{\bar{\alpha}}$, we only need to show that $(\mathcal{G}_c)_{\bar{0}}$ is generated by spaces $(\mathcal{G}_c)_{\bar{\alpha}}$, $\bar{\alpha} \neq \bar{0}$. By (1.41), it is enough to show that if $\bar{\beta} = \bar{0}$, then $\bar{\mathcal{L}}_{\beta}$ is generated by spaces $\mathcal{L}_{\bar{\alpha}}$, $\bar{\alpha} \neq 0$. So let $\bar{\beta} \neq 0$ and $\bar{x} \in \bar{\mathcal{L}}_{\beta}$, $x \in \mathcal{L}_{\beta} \cap \mathcal{L}_c$. Since $\mathcal{L}_c$ is generated by root space $\mathcal{L}_{\alpha}$, α not isotropic, so $\bar{x}$ is generated by spaces $\bar{\mathcal{L}}_{\alpha}$, α not isotropic. But $\bar{\mathcal{L}}_{\alpha} \subseteq \mathcal{L}_{\bar{\alpha}}$ and $\bar{\alpha} \neq 0$ by (1.39). Thus $\bar{x}$ is generated by spaces $\mathcal{L}_{\bar{\alpha}}$, $\bar{\alpha} \neq \bar{0}$. Hence (C12) holds. This completes the proof. □

2 Twisting by Automorphism

In this section, we briefly recall from [AABGP, III], the definition and some properties of a quantum torus with an involution (See Chapter 4, Section 6 of [M] for information about the quantum torus). All of our examples in this chapter will be subalgebras of a matrix algebra with coordinates from a quantum torus. Corresponding to any quantum torus there exists a semilattice which we will need here. Finally we introduce two tame generalized loop algebras which we will use in future sections.

Let $\mathbf{e} = (e_1, \ldots, e_\nu)$ be a vector in $\mathbb{C}^\nu$ and let $\mathbf{q} = (q_{ij})$ be a $\nu \times \nu$-matrix so that

$$e_i = \pm 1,\ 1 \leq i \leq \nu, \quad q_{ii} = 1,\ 1 \leq i \leq \nu \quad \text{and} \quad q_{ij} = q_{ji} = \pm 1,\ 1 \leq i \neq j \leq \nu. \tag{2.1}$$

Let $\mathcal{A}$ be the associative algebra over $\mathbb{C}$ with generators x_i, x_i^{-1}, $i = 1, \ldots, \nu$, subject to the relations

$$x_i x_j = q_{ij} x_j x_i, \quad 1 \leq i \neq j \leq \nu. \tag{2.2}$$

We use the notation $x^\sigma = x_1^{n_1} \cdots x_\nu^{n_\nu}$ for $\sigma = (n_1, \cdots, n_\nu) \in \mathbb{Z}^\nu$, and so we have $\mathcal{A} = \sum_{\sigma \in \mathbb{Z}^\nu} \mathbb{C} x^\sigma$. Let $^-$ be the involution (a period 2 anti-automorphism) on $\mathcal{A}$ such that

$$\bar{x}_i = e_i x_i, \quad i = 1, \ldots, \nu. \tag{2.3}$$

(It is easy to see, using (2.1), that the involution $^-$ satisfying (2.3) exists.) It is clearly unique. The associative algebra $(\mathcal{A}, ^-)$ is called the *quantum torus with involution determined by the vector* $\mathbf{e}$ *and the matrix* $\mathbf{q}$. With respect to the involution $^-$ we have $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, where

$$\mathcal{A}_+ = \{h \in \mathcal{A} \mid \bar{h} = h\} \quad \text{and} \quad \mathcal{A}_- = \{s \in \mathcal{A} \mid \bar{s} = -s\}. \tag{2.4}$$

Let

$$I_{\mathbf{e}} = \{i \mid 1 \leq i \leq \nu,\ e_i = -1\} \quad \text{and} \quad J_{\mathbf{q}} = \{(i,j) \mid 1 \leq i < j \leq \nu,\ q_{ij} = -1\}.$$

Let

$$\Lambda = \mathbf{Z}^\nu \quad \text{and} \quad \bar{\Lambda} = \Lambda/2\Lambda = \mathbf{F}_2^\nu,$$

where $\mathbf{F}_2$ is the field of two elements. Let $\bar{\ } : \Lambda \to \bar{\Lambda}$ be the canonical map. Recall that a map $Q : \bar{\Lambda} \to \mathbf{F}_2$ is a quadratic form on $\bar{\Lambda}$ if and only if satifies the conditions,

$$\begin{aligned} &\bullet\ Q(c\bar{\sigma}) = c^2 Q(\bar{\sigma}) \quad \text{for all } \sigma \in \Lambda,\ c \in \mathbf{F}_2 \\ &\bullet\ f(\bar{\sigma}, \bar{\eta}) := Q(\bar{\sigma} + \bar{\eta}) - Q(\bar{\sigma}) - Q(\bar{\eta}) \text{ is bilinear }, \sigma, \eta \in \Lambda, \end{aligned} \tag{2.5}$$

or equivalently, Q is a quadratic form on $\bar{\Lambda}$ if and only if it is of the form

$$Q((n_1, \ldots, n_\nu)) = \sum_{i \in I} n_i^2 + \sum_{(i,j) \in J} n_i n_j,$$

where I is a subset of $\{1, \ldots \nu\}$ and J is a subset of $\{(i,j) \mid 1 \leq i < j \leq \nu\}$. Put

$$Z(Q) = \{\sigma \in \Lambda \mid Q(\bar{\sigma}) = 0\}.$$

It is clear that $S = Z(Q)$ satisfies conditions (S1) through (S4) of Definition I.1.23 and hence $Z(Q)$ is a semilattice in $\mathbf{R}^\nu$. We denote the complement of a set Z in $\mathbf{Z}^\nu$ by Z^c.

Lemma 2.6 *Let Q be a quadratic form on $\bar{\Lambda}$ and let $0 \neq \tau \in Z(Q)^c$. Then there exists a quadratic form Q_τ on $\bar{\Lambda}$ so that*

$$Z(Q_\tau) = Z(Q)^c + \tau.$$

Proof. Since $\tau \in Z(Q)^c$, $Q(\bar{\tau}) = 1$. Therefore

$$0 = Q(2\bar{\tau}) = Q(\bar{\tau}) + Q(\bar{\tau}) + f(\bar{\tau}, \bar{\tau}) = 2 + f(\bar{\tau}, \bar{\tau}) = f(\bar{\tau}, \bar{\tau}),$$

and so

$$f(\bar{\tau}, \bar{\tau}) = 0.$$

We define

$$Q_\tau : \bar{\Lambda} \longrightarrow \mathsf{F}_2 \quad \text{by}$$
$$Q_\tau(\bar{\sigma}) = Q(\bar{\sigma}) + f(\bar{\sigma}, \bar{\tau}).$$

We show that (2.5) holds for Q_τ. For $c \in \mathsf{F}_2$ and $\sigma \in \Lambda$ we have

$$Q_\tau(c\bar{\sigma}) = Q(c\bar{\sigma}) + f(c\bar{\sigma}, \bar{\tau}) = c^2(Q(\bar{\sigma}) + f(\bar{\sigma}, \bar{\tau})) = c^2 Q_\tau(\bar{\sigma}).$$

(Here we used the fact that $c^2 = c$ for all $c \in \mathsf{F}_2$.) Also for $\sigma, \tau \in \Lambda$ we have

$$\begin{aligned} Q_\tau(\bar{\sigma} + \bar{\eta}) &= Q(\bar{\sigma} + \bar{\eta}) + f(\bar{\sigma} + \bar{\eta}, \bar{\tau}) \\ &= Q(\bar{\sigma}) + Q(\bar{\eta}) + f(\bar{\sigma}, \bar{\eta}) + f(\bar{\sigma}, \bar{\tau}) + f(\bar{\eta}, \bar{\tau}) \\ &= Q_\tau(\bar{\sigma}) + Q_\tau(\bar{\eta}) + f(\bar{\sigma}, \bar{\eta}). \end{aligned}$$

Thus (2.5) holds for Q_τ and so it is a quadratic form on $\bar{\Lambda}$.

Next, we show $Z(Q_\tau) = Z(Q)^c + \tau$. Let $\sigma \in \Lambda$, then

$$\begin{aligned} Q_\tau(\bar{\sigma}) &= Q(\bar{\sigma}) + f(\bar{\sigma}, \bar{\tau}) \\ &= Q(\bar{\sigma}) + Q(\bar{\tau}) + f(\bar{\sigma}, \bar{\tau}) + Q(\bar{\tau}) \\ &= Q(\bar{\sigma} + \bar{\tau}) + 1. \end{aligned}$$

Thus

$$\sigma \in Z(Q_\tau) \Longleftrightarrow \sigma + \tau \in Z(Q)^c \Longleftrightarrow \sigma \in Z(Q)^c + \tau.$$

□

Now, using $I_{\mathbf{e}}$ and $J_{\mathbf{q}}$ we define a quadratic form $Q_{\mathbf{e},\mathbf{q}}$ by,

$$Q_{\mathbf{e},\mathbf{q}}((n_1, \ldots, n_\nu)) = \sum_{i \in I_{\mathbf{e}}} n_i^2 + \sum_{(i,j) \in J_{\mathbf{q}}} n_i n_j = \sum_{i \in I_{\mathbf{e}}} n_i + \sum_{(i,j) \in J_{\mathbf{q}}} n_i n_j.$$

(If $I_{\mathbf{e}} = \emptyset$ or $J_{\mathbf{q}} = \emptyset$ we interpret the corresponding sum as 0.) We note that any quadratic form on $\bar{\Lambda}$ is of this form for some $\mathbf{e}$ and $\mathbf{q}$. Put

$$Z_{\mathbf{e},\mathbf{q}} = \{\sigma \in \mathbf{Z}^\nu \mid Q_{\mathbf{e},\mathbf{q}}(\bar{\sigma}) = 0\} \quad \text{and} \quad Z^c_{\mathbf{e},\mathbf{q}} = \mathbf{Z}^\nu \setminus Z_{\mathbf{e},\mathbf{q}}. \tag{2.7}$$

Then, $Z_{\mathbf{e},\mathbf{q}}$ is a semilattice in $\mathbf{R}^\nu$. By Lemma III.3.5 of [AABGP], we have

$$\begin{gathered} \mathcal{A}_+ = \operatorname{span}_{\mathbf{C}}\{x^\sigma \mid \sigma \in Z_{\mathbf{e},\mathbf{q}}\}, \quad \mathcal{A}_- = \operatorname{span}_{\mathbf{C}}\{x^\sigma \mid \sigma \in Z^c_{\mathbf{e},\mathbf{q}}\}, \quad \text{and} \\ \text{the involution } ^- = I \quad \Longleftrightarrow \quad \mathbf{e} = \mathbf{1}_\nu \text{ and } \mathbf{q} = \mathbf{1}_{\nu\times\nu}, \end{gathered} \tag{2.8}$$

(in which case $\mathcal{A}$ is the commutative ring of Laurent polynomials in ν variables.)

Next, suppose $n \geq 2$ and consider the associative matrix algebra $M_n(\mathcal{A})$. Define the $\mathbf{C}$–linear map $\epsilon : \mathcal{A} \longrightarrow \mathbf{C}$ by linear extension of

$$\epsilon(x^\sigma) = \begin{cases} 1 & \text{if } \sigma = 0 \\ 0 & \text{if } \sigma \neq 0. \end{cases}$$

Then the form $\epsilon(a, b) = \epsilon(ab)$ is a nondegenerate symmetric bilinear form on $\mathcal{A}$ preserved by $^-$; that is $\epsilon(\bar{a}, \bar{b}) = \epsilon(a, b)$ for all $a, b \in \mathcal{A}$. Also

$$\epsilon(tr(A)) = \epsilon(tr(\bar{A})) \quad \text{and} \quad \epsilon(tr(AB)) = \epsilon(tr(BA)), \tag{2.9}$$

for $A, B \in M_n(\mathcal{A})$. Define a form $(\cdot,\cdot)$ on $M_n(\mathcal{A})$ by

$$(A, B) = \epsilon(tr(AB)). \tag{2.10}$$

By (2.9), this form is an associative symmetric bilinear form on $M_n(\mathcal{A})$ which, as one can see by a straightforward computation, is nondegenerate. It also follows that $(\cdot,\cdot)$ is a nondegenerate invariant symmetric bilinear form on $gl_n(\mathcal{A})$, where $gl_n(\mathcal{A})$ is the Lie algebra with the underlying space $M_n(\mathcal{A})$ and the commutator product.

Next, suppose $l \geq 1$. Let

$$sl_{\ell+1}(\mathcal{A}) = \{Y \in M_{\ell+1}(\mathcal{A}) \mid tr(Y) \equiv 0 \bmod [\mathcal{A}, \mathcal{A}]\}. \tag{2.11}$$

Then $sl_{\ell+1}(\mathcal{A})$ is a Lie subalgebra of $gl_{\ell+1}(\mathcal{A})$. Let

$$\dot{\mathcal{H}} = \{\sum_{i=1}^{l} a_i(e_{ii} - e_{i+1,i+1}) \mid a_i \in \mathbf{C}\}.$$

Define

$$\deg(x^\sigma e_{ij}) = \sigma.$$

Then it is straightforward to see that properties (C1)-(C12) from Section 1 hold for triple $(sl_{\ell+1}(\mathcal{A}), (\cdot,\cdot), \dot{\mathcal{H}})$. Hence

$$\text{the triple } (sl_{\ell+1}(\mathcal{A}), (\cdot,\cdot), \dot{\mathcal{H}}) \text{ is a tame generalized loop algebra ot type } A_\ell. \tag{2.12}$$

(If $\mathcal{A}$ is commutative, (2.12) also follows from Example 1.29 of [AABGP. III]. In general if $\mathcal{G} = sl_{\ell+1}(\mathcal{A})$ and $\mathcal{L} = \mathcal{G} + \mathcal{C} + \mathcal{D}$ as in Section 1. then by [BGK] $\mathcal{L}$ is a nondegenerate EALA. Then it follows from Proposition 1.35 that $\mathcal{G}$ is a (tame) generalized loop algebra.)

Now let $\mathbf{e} = \mathbf{1}_\nu$ and $\mathbf{q} = \mathbf{1}_{\nu\times\nu}$. Then the quantum torus $(\mathcal{A}, ^-)$ is the commutative ring of Laurent polynomials in ν variables. Let

$$D_\ell(\mathcal{A}) = \{Y = \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \in M_{2\ell}(\mathcal{A}) \mid B^t = -B,\ C^t = -C,\ A, B, C \in M_\ell(\mathcal{A})\}. \tag{2.13}$$

Then $D_\ell(\mathcal{A})$ is a Lie subalgebra of $sl_{2\ell}(\mathcal{A})$. Indeed, $D_\ell(\mathcal{A})$ is the fixed point subalgebra of $sl_{2\ell}(\mathcal{A})$ with respect to the period 2−automorphism

$$\psi_K : sl_{2\ell}(\mathcal{A}) \longrightarrow sl_{2\ell}(\mathcal{A}), \text{ defined by}$$
$$\psi(Y) = -KY^tK,$$

where $K = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$. It is easy to see from (2.9) that the form $(\cdot,\cdot)$ restricted to $sl_{2\ell}(\mathcal{A})$ is invariant under the automorphism ψ_K. Then it follows from the nondegeneracy of the form on $sl_{2\ell}(\mathcal{A})$ and the next Lemma that

$$\text{the restriction of the form } (\cdot,\cdot) \text{ to } D_\ell(\mathcal{A}) \text{ is nondegenerate.} \tag{2.14}$$

Lemma 2.15 *Let $\mathcal{G}$ be an algebra with a nondegenerate form $(\cdot,\cdot)$ which is invariant under a period 2−automorphism ψ of $\mathcal{G}$. Then the restriction of the form to the fixed point subalgebra of $\mathcal{G}$, with respect to ψ, is also nondegenerate.*

Proof. Let X be in the radical of the restriction of the form to the fixed point subalgebra. Then for any $Y \in \mathcal{G}$, we have

$$0 = (Y + \psi(Y), X) = (Y, X) + (\psi(Y), X) = (Y, X) + (Y, \psi(X)) = 2(Y, X).$$

Thus X is in the radical of the form on $\mathcal{G}$ and so is the zero element. □

From [BGK, (2.19)] we have $D_\ell(\mathcal{A}) \cong \dot{\mathcal{G}} \otimes \mathcal{A}$ where $\dot{\mathcal{G}}$ is a finite dimensional simple Lie algebra of type D_ℓ. Then by Example 1.29 of [AABGP. III], we have

$$\text{if } \ell \geq 4, \text{ then } (D_\ell, (\cdot,\cdot), \dot{\mathcal{H}}) \text{ is a tame generalized loop } D_\ell. \tag{2.16}$$

3 Types A_1, B and BC as Twisted Subalgebras of Type A

Let $\nu \geq 1$. Let $\mathbf{e}$ be a vector in $\mathbf{C}^\nu$ and $\mathbf{q}$ be a $\nu \times \nu$ matrix satisfying (2.1). Let $(\mathcal{A}, ^-)$ be the quantum torus determined by $\mathbf{e}$ and $\mathbf{q}$.

Now suppose that $m \geq 1$ and $\tau_1, \ldots, \tau_m$ are elements of $\mathbf{Z}^\nu$ so that

$$\begin{gathered} \tau_1 = 0, \\ \tau_1, \ldots \tau_\nu \text{ represent distinct cosets of } 2\mathbf{Z}^\nu \text{ in } \mathbf{Z}^\nu, \quad \text{and} \\ \tau_i \in Z_{\mathbf{e},\mathbf{q}}, \quad i = 1, \ldots, m. \end{gathered} \tag{3.1}$$

Next let $l \geq 1$ and put

$$F = \begin{bmatrix} x^{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{\tau_m} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & I_\ell & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & F \end{bmatrix}.$$

Then F is an invertible $m \times m$-matrix and K is an invertible $n \times n$-matrix, where

$$n = 2\ell + m.$$

Put

$$\mathcal{G} = sl_n(\mathcal{A}).$$

Then by (2.12),

$$(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}}) \text{ is a tame generalized loop algebra of type } A_{n-1}.$$

By (3.1), we have $\bar{F}^t = F$ and $\bar{K}^t = K$. Note that for $Y = (y_{ij})$ and $Z = (z_{ij})$ in $M_n(\mathcal{A})$, we have

$$tr[Y, Z] = \sum_{i,j} y_{ij} z_{ji} - \sum_{i,j} z_{ji} y_{ij} \equiv 0 \bmod [\mathcal{A}, \mathcal{A}]$$

and so

$$tr(K^{-1}\bar{Y}^tK) \equiv tr(\bar{Y}) \mod [\mathcal{A},\mathcal{A}]. \tag{3.2}$$

Therefore, if $Y \in sl_n(\mathcal{A})$, then $K^{-1}\bar{Y}^tK \in sl_n(\mathcal{A})$. Define

$$\begin{aligned} \psi_K : \mathcal{G} &\longrightarrow \mathcal{G} \text{ by,} \\ \psi_K(Y) &= -K^{-1}\bar{Y}^tK. \end{aligned} \tag{3.3}$$

Since $\bar{K}^t = K$ and $\overline{YZ}^t = \bar{Z}^t\bar{Y}^t$ for $Y, Z \in M_n(\mathcal{A})$, it can be seen easily that ψ_K is a period 2–automorphism of $\mathcal{G}$. Denote by $\mathcal{G}_0$, the fixed point subalgebra of $\mathcal{G}$ with respect to ψ_K, that is

$$\mathcal{G}_0 = \{X = \psi_K(Y) + Y \mid Y \in \mathcal{G}\}.$$

The general form of a matrix in $\mathcal{G}_0$ is

$$X = \begin{bmatrix} A & S & -\bar{D}^tF \\ T & -\bar{A}^t & -\bar{C}^tF \\ C & D & B \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \bar{S}^t = -S, \quad \bar{T}^t = -T, \\ F^{-1}\bar{B}^tF = -B \quad \text{and} \\ tr(X) \equiv 0 \bmod [\mathcal{A},\mathcal{A}], \end{array} \tag{3.4}$$

where $A, S, T \in M_\ell(\mathcal{A})$, $C, D \in M_{m \times l}(\mathcal{A})$ and $B \in M_m(\mathcal{A})$.

On the other hand, if X is any matrix in $M_n(\mathcal{A})$ of the form (3.4), then $X = \psi(Y) + Y$, where

$$Y = \begin{bmatrix} A/2 & S/2 & 0 \\ T/2 & -\bar{A}^t/2 & 0 \\ C & D & B/2 \end{bmatrix}.$$

Thus

$$\mathcal{G}_0 \text{ consists of all matrices in } M_n(\mathcal{A}) \text{ of the form (3.4).} \tag{3.5}$$

Now let

$$\dot{\mathcal{H}}_0 = \{\sum_{i=1}^{l} a_i(e_{ii} - e_{\ell+i,\ell+i}) \mid a_i \in \mathbb{C}\},$$

and let $(\cdot,\cdot)$ be the form $(\cdot,\cdot)$ on $M_n(\mathcal{A})$, defined by (2.10), restricted to $\mathcal{G}_0$. It is easy to see that $(\cdot,\cdot)$ is invariant under ψ_K. Thus by Lemma 2.15, the restriction of the form $(\cdot,\cdot)$ on $\mathcal{G}_0$ is nondegenerate. For $\dot{\alpha} \in \dot{\mathcal{H}}_0^*$, let

$$(\mathcal{G}_0)_{\dot{\alpha}} = \{x \in \mathcal{G}_0 \mid [h, x] = \dot{\alpha}(h) \text{ for all } h \in \dot{\mathcal{H}}_0\}$$

and

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{H}}_0^* \mid (\mathcal{G}_0)_{\dot{\alpha}} \neq \{0\}\}.$$

It follows that $\dot{R}$ is an irreducible finite root system in the real span of $\dot{R}$. Moreover, if X is the type of $\dot{R}$, then

$$X = \begin{cases} A_1 & \text{if } l = 1,\ \mathbf{e} = \mathbf{1}_\nu \text{ and } \mathbf{q} = \mathbf{1}_{\nu\times\nu} \\ B_\ell & \text{if } l \geq 2,\ \mathbf{e} = \mathbf{1}_\nu \text{ and } \mathbf{q} = \mathbf{1}_{\nu\times\nu} \\ BC_\ell & \text{if } \mathbf{e} \neq \mathbf{1}_\nu \text{ or } \mathbf{q} \neq \mathbf{1}_{\nu\times\nu}. \end{cases} \tag{3.6}$$

(See [AABGP, III.3] for details.) Then by Proposition 3.14 and Lemma 3.31 of [AABGP, III] and (3.6) we have that

$$\text{the triple } (\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0) \text{ is a tame generalized loop algebra of type } X. \tag{3.7}$$

Moreover, we have the following result.

Theorem 3.8 *[AABGP, III.3.32] Let $\mathcal{G}_0$ be as above and let $\mathcal{L} = \mathcal{G}_0 \oplus \mathcal{C} \oplus \mathcal{D}$ be the Lie algebra consructed in Section 1(with $\mathcal{G}_0$ in place of $\mathcal{G}$). Then $\mathcal{L}$ is a tame EALA of nullity ν with root system*

$$R \cong \begin{cases} R(A_1, S) & \text{if } \ell = 1,\ \mathbf{e} = \mathbf{1} \text{ and } \mathbf{q} = \mathbf{1} \\ R(B_\ell, S, L) & \text{if } \ell \geq 2,\ \mathbf{e} = \mathbf{1} \text{ and } \mathbf{q} = \mathbf{1} \\ R(BC_1, S, E) & \text{if } \ell = 1, \text{ and } \mathbf{e} \neq \mathbf{1} \text{ or } \mathbf{q} \neq \mathbf{1} \\ R(BC_\ell, S, L, E) & \text{if } \ell \geq 2, \text{ and } \mathbf{e} \neq \mathbf{1} \text{ or } \mathbf{q} \neq \mathbf{1} \end{cases}.$$

where

$$S = \cup_{i=1}^m (\tau_i + 2\mathbf{Z}^\nu), \quad L = 2\mathbf{Z}^\nu \text{ and } \quad E = 2Z^c_{\mathbf{e},\mathbf{q}}.$$

4 Type C as a Twisted Subalgebra of Type A

As in Section 2, let $(\mathcal{A}, ^-)$ be the quantum torus determined by a vector $\mathbf{e}$ and a matrix $\mathbf{q}$, where $\mathbf{e}$ and $\mathbf{q}$ satisfy (2.1).

Suppose $l \geq 2$. Let

$$\mathcal{G} = sl_{2\ell}(\mathcal{A}),$$

defined by (2.11). Then by (2.12),

$$(\mathcal{G}, (\cdot,\cdot), \dot{\mathcal{H}}) \text{ is a tame generalized loop algebra of type } A_{2\ell-1}.$$

Let $K = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix}$ and define

$$\psi_K : \mathcal{G} \longrightarrow \mathcal{G} \text{ by}$$
$$\psi_K(Y) = -K^{-1}\bar{Y}^t K.$$

It is easy to see that ψ_K defines a period 2−automorphism of the Lie algebra $\mathcal{G}$. Denote by $\mathcal{G}_0$, the fixed point subalgebra of $\mathcal{G}$ with respect to ψ_K. A general form of a matrix in $\mathcal{G}_0$ is

$$X = \begin{bmatrix} A & B \\ C & -\bar{A}^t \end{bmatrix} \text{ with } \begin{array}{l} \bar{B}^t = B, \quad \bar{C}^t = C \quad \text{and} \\ tr(X) \equiv 0 \bmod [\mathcal{A}, \mathcal{A}], \end{array} \tag{4.1}$$

where $A, B, C \in M_\ell(\mathcal{A})$. On the other hand if $X \in M_{2\ell}(\mathcal{A})$ has the form (4.1), then $X = \psi_K(Y) + Y$, where $Y = \begin{bmatrix} A/2 & B/2 \\ C/2 & -\bar{A}^t/2 \end{bmatrix}$. Hence

$$\mathcal{G}_0 \text{ consists of all matrices in} M_{2\ell}(\mathcal{A}) \text{ of the form (4.1).} \tag{4.2}$$

Let

$$\dot{\mathcal{H}}_0 = \{\sum_{i=1}^{l} a_i(e_{ii} - e_{\ell+i,\ell+i}) \mid a_i \in \mathbb{C}\}$$

and let $(\cdot,\cdot)$ be the form $(\cdot,\cdot)$ on $M_{2n}(\mathcal{A})$ restricted to $\mathcal{G}_0$. It is easy to see that $(\cdot,\cdot)$ is invariant under ψ_K. Therefore by Lemma 2.15, the restriction of the form $(\cdot,\cdot)$ on $\mathcal{G}_0$ is nondegenerate. As usual for $\dot{\alpha} \in \dot{\mathcal{H}}_0^*$, let

$$(\mathcal{G}_0)_{\dot{\alpha}} = \{x \in \mathcal{G}_0 \mid [h, x] = \dot{\alpha}(h) \text{ for all } h \in \dot{\mathcal{H}}_0\}$$

and

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{H}}_0^* \mid (\mathcal{G}_0)_{\dot{\alpha}} \neq \{0\}\}.$$

By [AABGP, III.4], $\dot{R}$ is an irreducible finite root system of type C_ℓ in the real span of $\dot{R}$; and

$$\text{the triple } (\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0) \text{ is a generalized loop algebra of type } C_\ell. \tag{4.3}$$

Moreover, we have the following result.

Theorem 4.4 *[AABGP, III.4.7] Let $\mathcal{G}_0$ be as above and let $\mathcal{L} = \mathcal{G}_0 \oplus \mathcal{C} \oplus \mathcal{D}$ be the Lie algebra constructed in Section 1 (with $\mathcal{G}_0$ in place of $\mathcal{G}$). Then $\mathcal{L}$ is a tame EALA of nullity ν with root system*

$$R \cong R(C_\ell, \mathbf{Z}^\nu, Z_{\mathbf{e},\mathbf{q}}).$$

5 Types A_1 and B as Twisted Subalgebras of Type D

Let $\nu \geq 1$ and let $\mathcal{A}$ be the associative commutative algebra of Laurent polynomials in ν variables $x_1, \ldots, x_\nu$.

Now let $l \geq 1$, $m \geq 1$ and $\tau_1, \ldots \tau_m$ be elements of $\mathbf{Z}^\nu$ so that

$$\begin{gathered} \tau_1 = 0 \quad \text{and} \\ \tau_1, \ldots, \tau_m \text{ represent distinct cosets of } 2\mathbf{Z}^\nu \text{ in } \mathbf{Z}^\nu. \end{gathered} \tag{5.1}$$

Put

$$F = \begin{bmatrix} x^{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{\tau_m} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} I_\ell & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & I_\ell & 0 \\ 0 & F^{-1} & 0 & 0 \end{bmatrix}.$$

Then F is an invertible $m \times m$-matrix and K an invertible $(2n \times 2n)$-matrix with $K^{-1} = K$, where

$$n = l + m.$$

Let

$$\mathcal{G} := D_n(\mathcal{A}),$$

where $D_n(\mathcal{A})$ is the Lie subalgebra of $sl_{2n}(\mathcal{A})$ defined by (2.13). Then by (2.16),

if $n \geq 4$, $(D_n(\mathcal{A}), (\cdot,\cdot), \dot{\mathcal{H}})$ is a tame generalized loop algebra of type D_n.

Define

$$\begin{gathered} \psi_K : \mathcal{G} \longrightarrow \mathcal{G} \text{ by,} \\ \psi_K(Y) = K^{-1} Y K = K Y K. \end{gathered} \tag{5.2}$$

Then, $\psi(K)$ is a period 2$-$automorphism of $\mathcal{G}$. Denote by $\mathcal{G}_0$, the fixed point subalgebra of $\mathcal{G}$ with respect to ψ_K. So

$$\mathcal{G}_0 = \{X = \psi_K(Y) + Y \mid Y \in \mathcal{G}\}.$$

A typical element of $\mathcal{G}$ has the form

$$Y = \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & -B_2^t & B_4 \\ C_1 & C_2 & -A_1^t & -A_3^t \\ -C_2^t & C_4 & -A_2^t & -A_4^t \end{bmatrix} \quad \text{with} \quad \begin{array}{ll} B_1^t = -B_1, & B_4^t = -B_4, \\ C_1^t = -C_1, & C_4^t = -C_4 \end{array} .$$

where $A_1, B_1, C_1 \in M_\ell(\mathcal{A})$, $A_4, B_4, C_4 \in M_m(\mathcal{A})$, $A_2, B_2, C_2 \in M_{l\times m}(\mathcal{A})$ and $A_3 \in M_{m\times l}$. If $X := \psi(Y) + Y$, then

$$X = \begin{bmatrix} 2A_1 & A_2 + B_2F^{-1} & 2B_1 & B_2 + A_2F \\ A_3 - FC_2^t & A_4 - FA_4^tF^{-1} & -B_2^t - FA_2^t & B_4 + FC_4F \\ 2C_1 & C_2 - A_3^tF^{-1} & -2A_1^t & -A_3^t + C_2F \\ -C_2^t + F^{-1}A_3 & C_4 + F^{-1}B_4F^{-1} & -A_2^t - F^{-1}B_2^t & -A_4^t + F^{-1}A_4F \end{bmatrix} .$$

Therefore, the general form of a matrix X in $\mathcal{G}_0$ is

$$X = \begin{bmatrix} A & -D^t & S & -D^tF \\ FC & -B^t & FD & FPF \\ T & -C^t & -A^t & -C^tF \\ C & P & D & B \end{bmatrix} \quad \text{with} \quad \begin{array}{c} S^t = -S,\ T^t = -T,\ P^t = -P \\ \text{and } F^{-1}B^tF = -B, \end{array} \tag{5.3}$$

where $A, S, T \in M_\ell(\mathcal{A})$, $D, C \in M_{m\times\ell}(\mathcal{A})$ and $B, P \in M_m(\mathcal{A})$. Let us denote a matrix of the form (5.3) by

$$X = X(A, B, C, D, P, S, T).$$

If $X = X(A, B, C, D, P, S, T) \in M_{2n}(\mathcal{A})$ has the form (5.3) and we put

$$Y = \begin{bmatrix} A/2 & 0 & S/2 & -D^tF \\ 0 & -B^t/2 & FD & FPF \\ T/2 & -C^t & -A^t/2 & 0 \\ C & 0 & 0 & B/2 \end{bmatrix}$$

Then $Y \in \mathcal{G}$ and $X = \psi(Y) + Y$. Thus

$$\mathcal{G}_0 \text{ consists of all matrices } X(A, B, C, D, P, S, T) \in \mathcal{G} \text{ of the form (5.3).} \tag{5.4}$$

Next, we want to introduce a form $(\cdot,\cdot)$ on $\mathcal{G}_0$ and a subalgebra $\dot{\mathcal{H}}_0$ of $\mathcal{G}_0$. Let $(\cdot,\cdot)$ be the form on $M_{2n}(\mathcal{A})$, defined by (2.10), restricted to $\mathcal{G}_0$. It is easy to see that the form $(\cdot,\cdot)$ is invariant under ψ_K. Then Lemma 2.15 gives that

$$\text{the form } (\cdot,\cdot) \text{ on } M_{2n}(\mathcal{A}) \text{ restricted to } \mathcal{G}_0 \text{ is nondegenerate.} \tag{5.5}$$

Next, we want to define a finite dimensional subalgebra $\dot{\mathcal{H}}_0$ of $\mathcal{G}_0$. Let

$$\dot{\mathcal{H}}_0 = \{\sum_{i=1}^{\ell} a_i(e_{i,i} - e_{\ell+m+i,\ell+m+i}) \mid a_1,\dots,a_\ell \in \mathbb{C}\}.$$

Then, $\dot{\mathcal{H}}_0$ is an abelian subalgebra of $\mathcal{G}$. We define $\epsilon_j \in \dot{\mathcal{H}}_0^*$, by

$$\epsilon_j(e_{ii} - e_{\ell+m+i,\ell+m+i}) = \delta_{ij}, \quad \text{for} \quad 1 \le i,j \le \ell.$$

Let

$$(\mathcal{G}_0)_{\dot\alpha} = \{x \in \mathcal{G}_0 \mid [h,x] = \dot\alpha(h)x \text{ for all } h \in \dot{\mathcal{H}}_0\} \quad (\dot\alpha \in \dot{\mathcal{H}}_0^*).$$

Then

$$\begin{aligned} \mathcal{G}_0 = \sum_{\dot\alpha \in \dot{\mathcal{H}}_0^*} (\mathcal{G}_0)_{\dot\alpha} = (\mathcal{G}_0)_0 + \sum_{1\le i\ne j\le \ell} (\mathcal{G}_0)_{\epsilon_i-\epsilon_j} + \sum_{1\le i<j\le\ell} \Big((\mathcal{G}_0)_{\epsilon_i+\epsilon_j} + (\mathcal{G}_0)_{-\epsilon_i-\epsilon_j}\Big) \\ + \sum_{1\le i\le \ell}\Big((\mathcal{G}_0)_{\epsilon_i} + (\mathcal{G}_0)_{-\epsilon_i}\Big), \end{aligned} \tag{5.6}$$

where

$$(\mathcal{G}_0)_0 \text{ consists of all matrices } X(A,B,0,0,P,0,0) \in \mathcal{G} \text{ in which } A \text{ is diagonal,} \tag{5.7}$$

$$(\mathcal{G}_0)_{\epsilon_i-\epsilon_j} = \{ae_{i,j} - ae_{\ell+m+j,\ell+m+i} \mid a \in \mathcal{A}\}, \tag{5.8}$$

$$(\mathcal{G}_0)_{\epsilon_i+\epsilon_j} = \{ae_{i,\ell+m+j} - ae_{j,\ell+m+i} \mid a \in \mathcal{A}\}, \tag{5.9}$$

$$(\mathcal{G}_0)_{-\epsilon_i-\epsilon_j} = \{ae_{\ell+m+i,j} - ae_{j,\ell+m+i} \mid a \in \mathcal{A}\}, \tag{5.10}$$

$$(\mathcal{G}_0)_{\epsilon_i} = \{\sum_{j=1}^{m} a_j(e_{2\ell+m+j,\ell+m+i} - e_{i,\ell+j} - x^{\tau_j}e_{i,2\ell+m+j} + x^{\tau_j}e_{\ell+j,\ell+m+i}) \mid a_j \in \mathcal{A}\} \text{ and} \tag{5.11}$$

$$(\mathcal{G}_0)_{-\epsilon_i} = \{\sum_{j=1}^{m} a_j(e_{2\ell+m+j,i} - x^{\tau_j}e_{\ell+j,i} - e_{\ell+m+i,\ell+j} - x^{\tau_j}e_{\ell+m+i,2\ell+m+j}) \mid a_j \in \mathcal{A}\}. \tag{5.12}$$

Let $\dot R = \{\dot\alpha \in \dot{\mathcal{H}}_0^* \mid (\mathcal{G}_0)_{\dot\alpha} \ne 0\}$. Then

$$\dot R = \{0\} \cup \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i < j \le \ell\} \cup \{\pm\epsilon_i \mid 1 \le i \le \ell\}.$$

Using (2.10), one can see that

$$(\epsilon_i, \epsilon_j) = \frac{1}{2}\delta_{ij}. \tag{5.13}$$

Hence

$$\dot{R} \text{ is an irreducible finite root system of type } X \text{ in } \dot{V} = \sum_{i=1}^{l} \mathbf{R}\epsilon_i, \text{ where} \tag{5.14}$$

$$X = \begin{cases} A_1 & \text{if } l = 1 \\ B_\ell & \text{if } l \geq 2. \end{cases}$$

Next, we want to give a $\mathbf{Z}^\nu$-grading to $\mathcal{G}_0$. We first give a $\mathbf{Z}^\nu$-grading to $M_{2n}(\mathcal{A})$ by defining

$$\deg(x^\sigma e_{pq}) = 2\sigma + \lambda_p - \lambda_q, \tag{5.15}$$

where λ_i's are defined by

$$\begin{gathered} \lambda_1 = 0, \ldots, \lambda_\ell = 0, \qquad \lambda_{\ell+1} = -\tau_1, \ldots, \lambda_{\ell+m} = -\tau_m, \\ \lambda_{\ell+m+1} = 0, \ldots, \lambda_{2\ell+m} = 0 \quad \text{and} \quad \lambda_{2\ell+m+1} = \tau_1, \ldots, \lambda_{2\ell+2m} = \tau_m. \end{gathered} \tag{5.16}$$

Then $M_{2n}(\mathcal{A})$ and consequently $gl_{2n}(\mathcal{A})$ become $\mathbf{Z}^\nu$-graded Lie algebras with $gl_{2n}(\mathcal{A}) = M_{2n}(\mathcal{A})^\sigma$. To show $\mathcal{G}_0$ is also $\mathbf{Z}^\nu-$graded with respect to the above grading, we need to show that $\mathcal{G}_0$ is generated as a vector space over $\mathbf{C}$ by elements which are homogenous with respect to the $\mathbf{Z}^\nu-$grading on $\mathcal{G}$. First, note that as a vector space

$(\mathcal{G}_0)_0$ is generated by elements which have one of the forms,

(1) $x^\sigma(e_{i,i} - e_{\ell+m+i,\ell+m+i}),\ 1 \leq i \leq \ell, \sigma \in \mathbf{Z}^\nu$

(2) $(x^\sigma x^{\tau_j} e_{2\ell+m+i,2\ell+m+j} - x^\sigma x^{\tau_i} e_{2\ell+m+j,2\ell+m+i})$
$\quad - (x^\sigma x^{\tau_j} e_{\ell+j,\ell+i} - x^\sigma x^{\tau_i} e_{\ell+i,\ell+j}),\ 1 \leq i < j \leq m, \sigma \in \mathbf{Z}^\nu$

(3) $x^\sigma(e_{2\ell+m+i,\ell+j} - e_{2\ell+m+j,\ell+i}) + x^\sigma x^{\tau_i+\tau_j}(e_{\ell+i,2\ell+m+j} - e_{\ell+j,2\ell+m+i}),\ (i \neq j),$
$\quad 1 \leq i < j \leq m,\ \sigma \in \mathbf{Z}^\nu.$

(5.17)

Note that elements of the form (1) generate all matrices of the form $X(A, 0, 0, 0, 0, 0, 0)$, A diagonal, elements of the form (2) generate all matrices of the form $X(0, B, 0, 0, 0, 0, 0)$ (since $F^{-1}B^tF = -B$) and elements of the form (3) generate all matrices of the form $X(0, 0, 0, 0, P, 0, 0)$ (since $P^t = -P$). One can check that,

$$\begin{gathered} \text{each element of the form (1) has degree } 2\sigma \text{ and} \\ \text{each element of the form (2) or (3) has degree } 2\sigma + \tau_i + \tau_j. \end{gathered} \tag{5.18}$$

From (5.8)-(5.10) we see that

$$\begin{gathered}\text{each of the spaces } (\mathcal{G}_0)_{\epsilon_i-\epsilon_j},\ (\mathcal{G}_0)_{\epsilon_i+\epsilon_j} \text{ and } (\mathcal{G}_0)_{-\epsilon_i-\epsilon_j}, \\ \text{is generated by elements of degree } 2\sigma \text{ for some } \sigma \in \mathbf{Z}^\nu.\end{gathered} \tag{5.19}$$

Also from (5.11) and (5.12), we see that

$$\begin{gathered}\text{each of the spaces } (\mathcal{G}_0)_{\epsilon_i} \text{ and } (\mathcal{G}_0)_{-\epsilon_i} \text{ is generated by elements of degree} \\ 2\sigma + \tau_j,\ 1 \le j \le m.\end{gathered} \tag{5.20}$$

By (5.6), (5.17), (5.18), (5.19) and (5.20), $\mathcal{G}_0$ is generated by homogeneous elements with respect to the $\mathbf{Z}^\nu$-grading on $gl_{2n}(\mathcal{A})$. Hence,

$$\mathcal{G}_0 \text{ is } \mathbf{Z}^\nu\text{-graded with } \mathcal{G}_0 = \sum_{\sigma\in\mathbf{Z}^\nu} \mathcal{G}_0^\sigma, \text{ where } \mathcal{G}_0^\sigma = \mathcal{G}_0 \cap gl_{2n}(\mathcal{A})^\sigma. \tag{5.21}$$

Moreover, as we have just seen

$$(\mathcal{G}_0)_{\dot\alpha} = \sum_{\sigma\in\mathbf{Z}^\nu} \mathcal{G}_0^\sigma \cap (\mathcal{G}_0)_{\dot\alpha} \text{ for all } \dot\alpha \in \dot R. \tag{5.22}$$

Lemma 5.23 *The Lie algebra $(\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ satisfies conditions (C1) through (C11) and so is a generalized loop algebra.*

Proof. (C1), (C2), (C3), (C4), (C5) and (C6) follow respectively by (5.5), (5.6), (5.13), (5.6), (5.21) and (5.22). (C7) follows from (5.17), (5.18) and the fact that if $i \neq j$, then $2\sigma+\tau_i+\tau_j \neq 0$ for all $\sigma \in \mathbf{Z}^\nu$ (see (5.1)). By (5.18), $\mathcal{G}_0^{2\sigma} \neq 0$ for all $\sigma \in \mathbf{Z}^\nu$. Thus (C8) holds. To see that (C9) holds, it is suffices to show that $(M_{2n}(\mathcal{A})^\sigma, M_{2n}(\mathcal{A})^\tau) = \{0\}$ if $\sigma, \tau \in \mathbf{Z}^\nu$ with $\sigma + \tau \neq 0$. But for $1 \le i, j, p, q \le 2n$ and $\sigma, \tau \in \mathbf{Z}^\nu$ we have

$$(x^\sigma e_{ij}, x^\tau e_{pq}) = \epsilon(tr(x^\sigma e_{ij} x^\tau e_{pq})) = \delta_{jp}\delta_{iq}\delta_{\sigma,-\tau}$$

which is zero if $2\sigma + \lambda_i - \lambda_j + 2\tau + \lambda_p - \lambda_q \neq 0$. Thus (C9) holds.

Next, we want to find an element $\tau \in \mathbf{Z}^\nu$ such that (C10) holds. Let $\tau = 0$. Then (C10) holds by (5.8) through (5.12), (5.19), (5.20) and the fact that $\tau_1 = 0$. Finally, we show that (C11) holds. Let $\sigma \in R^0$. Thus $\mathcal{G}_0 \cap \mathcal{G}_0^\sigma \neq \{0\}$. By (5.17) and (5.18), we have $\sigma = 2\xi + \tau_i + \tau_j$ for some $\xi \in \mathbf{Z}^\nu$ and $1 \le i, j \le m$. Let $\eta := -2\xi - \tau_i$. Then $\sigma + \eta = \tau_j$. Now by (5.11) and (5.20), we have

$$(\mathcal{G}_0)_{\epsilon_1} \cap \mathcal{G}_0^\eta \neq \{0\} \quad \text{and} \quad (\mathcal{G}_0)_{\epsilon_1} \cap \mathcal{G}_0^{\sigma+\eta} \neq \{0\}.$$

Hence (C11) holds. This finishes the proof of Lemma. □

To see if the condition (C12) holds for $\mathcal{G}_0$, we need to compute the subalgebra $(\mathcal{G}_0)_c$ of $\mathcal{G}_0$ (see Definition 1.20).

Lemma 5.24 *We have*

$$(\mathcal{G}_0)_c = \{X(A,B,C,D,P,S,T) \in \mathcal{G}_0 \mid P = BF^{-1}\}. \tag{5.25}$$

Proof. Let us denote by M the right hand side of the equality in (5.25). Then M is a subalgebra of $\mathcal{G}_0$. Indeed, if $X_i := X(A_i, B_i, C_i, D_i, P_i, S_i, T_i) \in M$ with $P_i = B_i F^{-1}$ for $i = 1, 2$, then

$$[X_1, X_2] = X(*, B, *, *, P, *, *), \quad \text{where}$$

$B = -C_1 D_2^t F + P_1 F P_2 F - D_1 C_2^t F + B_1 B_2 + C_2 D_1^t F - P_2 F P_1 F + D_2 C_1^t F - B_2 B_1$ and $P = -C_1 D_2^t - B_1 F^{-1} B_2^t - D_1 C_2^t + B_1 B_2 F^{-1} + C_2 D_1^t + B_2 F^{-1} B_1^t + D_2 C_1^t - B_2 B_1 F^{-1}$. Since $F^{-1} B_i^t F = -B_i$ and $P_i = B_i F^{-1}$ for $i = 1, 2$, it follows that $P = BF^{-1}$. Thus $[X_1, X_2] \in M$. By (5.8)-(5.12), M contains all spaces $(\mathcal{G}_0)_{\dot\alpha}$, $\dot\alpha \in \dot{R}^\times$. Thus $(\mathcal{G}_0)_c \subseteq M$.

Conversely, we show that $M \subseteq (\mathcal{G}_0)_c$. Let $X = X(A, B, C, D, BF^{-1}, S, T) \in M$. We want to show $X \in (\mathcal{G}_0)_c$. By (5.9)-(5.12), $X(0, 0, C, D, 0, S, T) \in (\mathcal{G}_0)_c$. Subtracting this element from X, we can assume that $X = X(A, B, 0, 0, BF^{-1}, 0, 0)$. We are done if we show that $X_A := X(A, 0, 0, 0, 0, 0, 0)$ and $X_B := (0, B, 0, 0, BF^{-1}, 0, 0)$ are in $(\mathcal{G}_0)_c$. First, we show $X_A \in (\mathcal{G}_0)_c$. By (5.8), each matrix of the form $X(A_1, 0, 0, 0, 0, 0, 0)$ in which A_1 has zero diagonal is in $(\mathcal{G}_0)_c$. So subtracting an element of this form from X_A we can assume that A is a diagonal matrix. Consider two matrices in $(\mathcal{G}_0)_c$ of the form $X_{C_1} := (0, 0, C_1, 0, 0, 0, 0)$ and $X_{D_1} := (0, 0, 0, D_1, 0, 0, 0)$. Then

$$\begin{gathered}[X_{D_1}, X_{C_1}] = X(-2D_1^t F C_1, -D_1 C_1^t F + C_1 D_1^t F, 0, 0, -D_1 C_1^t + C_1 D_1^t, 0, 0) \in (\mathcal{G}_0)_c, \\ \text{for all } C_1, D_1 \in M_{m \times l}(\mathcal{A}).\end{gathered} \tag{5.26}$$

Fix $0 \neq a \in \mathcal{A}$ and take $D_1 = (d_{ij}) \in M_{m \times l}(\mathcal{A})$ to be the matrix whose only nonzero entry is $d_{11} = a \in \mathcal{A}$ and $C_1 = (c_{ij})$ to be the matrix whose only nonzero entry is $c_{11} = -1/2$. Then

$$[X_{D_1}, X_{C_1}] = ae_{11} - ae_{\ell+m+1, \ell+m+1} \in (\mathcal{G}_0)_c. \tag{5.27}$$

Now, consider two elements $X_1 = ae_{1j} - ae_{\ell+m+j,\ell+m+1}$ and $X_2 = e_{j1} - e_{\ell+m+1,\ell+m+j}$ in $(\mathcal{G}_0)_c$, $1 < j \leq \ell$. Then

$$[X_1, X_2] = (ae_{11} - ae_{\ell+m+1,\ell+m+1}) - (ae_{jj} - ae_{\ell+m+j,\ell+m+j}) \in (\mathcal{G}_0)_c. \tag{5.28}$$

Now (5.27) and (5.28) implies that $X_A \in (\mathcal{G}_0)_c$. Finally, we show that $X_B \in (\mathcal{G}_0)_c$.

First, note that since $F^{-1}B^tF = -B$, B is a sum of elements of the form $ax^{\tau_j}e_{ij} - ax^{\tau_i}e_{ji}$, $a \in \mathcal{A}$, $1 \leq i, j \leq m$. (In particular B has zero diagonal.) So it is enough to show that $X_B \in (\mathcal{G}_0)_c$ where $B = ax^{\tau_j}e_{ij} - ax^{\tau_i}e_{ji}$ for some $a \in \mathcal{A}$ and $1 \leq i \neq j \leq m$. Fix $a \in \mathcal{A}$ and $1 \leq i \neq j \leq m$. Take $D_1 = -ae_{i1}$ and $C_1 = e_{j1}$. Then $D_1^tFC_1 = 0$ and $-D_1C_1^tF + C_1D_1^tF = ax^{\tau_j}e_{ij} - ax^{\tau_i}e_{ji}$. From (5.26) we get that $X_B \in (\mathcal{G}_0)_c$. □

Corollary 5.29 *$(\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ is a tame generalized loop algebra if and only if $m = 1$.*

Proof. By Lemma 5.23, we only need to show that (C12) holds if and only if $m = 1$. This is equivalent to say $(\mathcal{G}_0)_0 \subseteq (\mathcal{G}_0)_c$ if and only if $m = 1$.

Consider a typical element $X(A, B, C, D, P, S, T) \in \mathcal{G}_0$. Since $B, P \in M_m(\mathcal{A})$, $P^t = -P$ and $F^{-1}B^tF = -B$, so if $m = 1$, then $B = 0$ and $P = 0$. Thus by (5.7),

$$(\mathcal{G}_0)_0 = \{X(A, 0, 0, 0, 0, 0, 0) \in \mathcal{G}_0 \mid A \text{ diagonal}\}.$$

Hence $(\mathcal{G}_0)_0 \subseteq (\mathcal{G}_0)_c$, by Lemma 5.24. If $m > 1$, then there exists $0 \neq P \in M_m(\mathcal{A})$ with $P^t = -P$. Thus $X(0, 0, 0, 0, P, 0, 0) \in (\mathcal{G}_0)_0$, by (5.7), and $X(0, 0, 0, 0, P, 0, 0) \notin (\mathcal{G}_0)_c$, by Lemma 5.24. Thus $(\mathcal{G}_0)_0 \not\subseteq (\mathcal{G}_0)_c$. □

Proposition 5.30 *Suppose $l \geq 1$, $m \geq 1$ and $\tau_1, \ldots, \tau_m$ satisfy (5.1). Let $(\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ be the triple defined above. Then the triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ constructed in Section 1 (with $\mathcal{G}_0$ and $\dot{\mathcal{H}}_0$ in place of $\mathcal{G}$ and $\dot{\mathcal{H}}$) is an EALA of nullity ν with root system*

$$R \cong \begin{cases} R(A_1, S) & \text{if } l = 1 \\ R(B_\ell, S, L) & \text{if } l \geq 2 \end{cases}$$

where $S = \cup_{i=1}^m (\tau_i + 2\mathbf{Z}^\nu)$ and $L = 2\mathbf{Z}^\nu$.

Proof. By Lemma (5.23), Corollary 5.29 and Proposition 1.21, $\mathcal{L}$ is an EALA of nullity ν. By (5.14) and Proposition 1.21, R is isomorphic to the root system indicated in the statement. Finally by (1.22), (5.8)-(5.12) and (5.19)-(5.20), we have $S = \cup_{i=1}^{m}(\tau_i + 2\mathbb{Z}^\nu)$ and $L = 2\mathbb{Z}^\nu$. $\square$

To see when the EALA $\mathcal{L}$ is tame we need to compute $\mathcal{L}_c^\perp$, the orthogonal complement of $\mathcal{L}_c$ in $\mathcal{L}$, defined by (I.1.41). We start by computing $(\mathcal{G}_0)_c^\perp$. By 1.32, we have

$$(\mathcal{G}_0)_c^\perp = \{x \in \mathcal{G}_0 \mid [x, (\mathcal{G}_0)_c] = \{0\}\}.$$

Lemma 5.31 $(\mathcal{G}_0)_c^\perp = \{X(0, B, 0, 0, P, 0, 0) \mid P = -BF^{-1}\}$.

Proof. First, we show that $(\mathcal{G}_0)_c^\perp$ is a subset of the right hand side of the equality in the statement. So let $X = X(A, B, C, D, P, S, T) \in (\mathcal{G}_0)_c^\perp$. By Lemma 5.24,

$$X_{A_1} := X(A_1, 0, 0, 0, 0, 0, 0) \in (\mathcal{G}_0)_c, \quad \text{for all matrices } A_1 \in M_\ell(\mathcal{A}).$$

Thus for all $A_1 \in M_\ell(\mathcal{A})$ we have

$$0 = [X, X_{A_1}] = X(AA_1 - A_1A, 0, CA_1, -DA_1^t, 0, -SA_1^t - A_1S, TA_1 + A_1^tT).$$

From this we get $C = 0$, $D = 0$, $S = 0$, $T = 0$ and $A = aI_\ell$ for some $a \in \mathcal{A}$. Thus $X = X(A, B, 0, 0, P, 0, 0)$ with $A = aI_\ell$. Next, let $D_1 \in M_{m\times l}$. Then $X_{D_1} = X(0, 0, 0, D_1, 0, 0, 0) \in (\mathcal{G}_0)_c$. Thus

$$0 = [X, X_{D_1}] = X(0, 0, 0, PFD_1 + BD_1 + D_1A^t, 0, 0, 0), \text{ for all matrices } D_1 \in M_{m\times l}(\mathcal{A}).$$

This implies that $PF + B = -aI_m$. Taking transpose of this equality and using $P^t = -P$ and $F^{-1}B^tF = -B$, we get $PF + B = aI_m$. Thus $a = 0$ and so $P = -BF^{-1}$. Hence X belongs to the right hand side of the equality in the statement. On the other hand one can easily see that for an arbitrary element $X(A_1, B_1, C_1, D_1, B_1F^{-1}, S_1, T_1) \in (\mathcal{G}_0)_c$, we have

$$[X(A_1, B_1, C_1, D_1, B_1F^{-1}, S_1, T_1), X(0, B, 0, 0, -BF^{-1}, 0, 0)] = 0.$$

Hence, the right hand side of the equality in the statement is a subset of $(\mathcal{G}_0)_c^\perp$. $\square$

Since $\mathcal{G}_0$ is a generalized loop algebra, from Corollary 1.30 we get

$$\mathcal{L}_c = (\mathcal{G}_0)_c \oplus \mathcal{C}. \tag{5.32}$$

Proposition 5.33 *The EALA $\mathcal{L}$ introduced in Proposition 5.30 is tame if and only if $m = 1$.*

Proof. We must show that $\mathcal{L}_c^{\perp} \subseteq \mathcal{L}_c$ if and only if $m = 1$. If $m = 1$, then by Lemma 5.29, $\mathcal{G}_0$ is a tame generalized loop algebra and so $\mathcal{L}$ is a tame EALA, by Proposition 1.21.

Now let $m > 1$. We have $(\mathcal{G}_0)_c^{\perp} \neq \{0\}$ in this case. By Lemmas 5.24 and 5.31, $(\mathcal{G}_0)_c^{\perp} \cap (\mathcal{G}_0)_c = \{0\}$ and so $(\mathcal{G}_0)_c^{\perp} \cap \mathcal{L}_c = \{0\}$ (since $\mathcal{L}_c = (\mathcal{G}_0)_c \oplus \mathcal{C}$.) By Lemma 1.33, $(\mathcal{G}_0)_c^{\perp} \subseteq \mathcal{L}_c^{\perp}$. Thus $\mathcal{L}_c^{\perp} \not\subseteq \mathcal{L}_c$. ◻

Next we would like to determine a subalgebra of $\mathcal{L}$ which is tame, regardless of m. For this purpose consider the triple $((\mathcal{G}_0)_c, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ where $(\cdot,\cdot)$ is the form $(\cdot,\cdot)$ on $\mathcal{G}_0$ restricted to $(\mathcal{G}_0)_c$. By Lemma 5.24, we have $\dot{\mathcal{H}}_0 \subseteq (\mathcal{G}_0)_c$ and by Lemma 5.24 and Lemma 5.31, $(\mathcal{G}_0)_c \cap (\mathcal{G}_0)_c^{\perp} = \{0\}$; that is the restriction of the form to $(\mathcal{G}_0)_c$ is nondegenerate. Therefore by Lemma 1.34, we have

$$((\mathcal{G}_0)_c, (\cdot,\cdot), \dot{\mathcal{H}}_0) \text{ is a tame generalized loop algebra.} \tag{5.34}$$

Hence using Proposition 1.21, we have the following result.

Proposition 5.35 *The Lie algebra $(\mathcal{G}_0)_c \oplus \mathcal{C} \oplus \mathcal{D}$ constructed in Section 1 (with $(\mathcal{G}_0)_c$ in place of $\mathcal{G}$) is a tame EALA of nullity ν and type X, where $X = A_1$ if $l = 1$ and $X = B_\ell$ if $l \geq 2$. Moreover, $(\mathcal{G}_0)_c \oplus \mathcal{C} \oplus \mathcal{D}$ has root system*

$$R \cong \begin{cases} R(A_1, S) & \text{if } l = 1 \\ R(B_\ell, S, L) & \text{if } l \geq 2 \end{cases}$$

where $S = \cup_{i=1}^{m}(\tau_i + 2\mathbf{Z}^{\nu})$ and $L = 2\mathbf{Z}^{\nu}$. ◻

We conclude this section with the following remark.

Remark 5.36 *The Lie algebra $(\mathcal{G}_0)_c$ is in fact isomorphic to the Lie algebra $\mathcal{G}_0$ of type A_1 or B_ℓ constructed in Section 3 where there $\mathcal{G}_0$ is the Lie subalgebra of $gl_{2\ell+m}(\mathcal{A})$ consisting of all matrices of the form*

$$Y = \begin{bmatrix} A & S & -D^tF \\ T & -A^t & -D^tC \\ C & D & B \end{bmatrix} \quad \text{with} \quad \begin{matrix} S^t = -S, \ T^t = -T \text{ and} \\ F^{-1}B^tF = -B. \end{matrix}$$

where $A, S, T \in M_\ell(\mathcal{A})$, $B \in M_m(\mathcal{A})$ and $C, D \in M_{l\times m}(\mathcal{A})$. This can be easily seen by observing that the assignment

$$Y = \begin{bmatrix} A & S & -D^tF \\ T & -A^t & -C^tF \\ C & D & B \end{bmatrix} \longmapsto X = \begin{bmatrix} A & -\frac{1}{2}D^t & \frac{1}{2}S & -\frac{1}{2}D^tF \\ FC & -\frac{1}{2}B^t & \frac{1}{2}FD & \frac{1}{2}FB \\ 2T & -C^t & -A^t & -C^tF \\ C & \frac{1}{2}BF^{-1} & \frac{1}{2}D & \frac{1}{2}B \end{bmatrix}$$

defines an isomorphism of $\mathcal{G}$ onto $(\mathcal{G}_0)_c$.

6 Type BC as a Twisted Subalgebra of Type C

Next, we want to show that EALA's of type BC can be constructed by twisting an EALA of type C.

Let $\mathbf{e}$ be a vector and $\mathbf{q}$ a matrix satisfying (2.1). Moreover assume that

$$\mathbf{e} \neq \mathbf{1}_\nu \quad \text{or} \quad \mathbf{q} \neq \mathbf{1}_{\nu\times\nu}. \tag{6.1}$$

Then by (2.8), $Z^c_{\mathbf{e},\mathbf{q}} \neq \emptyset$. Let $m \geq 1$ and $\tau_1, \ldots, \tau_m$ be elements of $\mathbf{Z}^\nu$ such that

$$\begin{gathered} \tau_i\text{'s represent distinct cosets of } 2\mathbf{Z}^\nu \text{ in } \mathbf{Z}^\nu \text{ and} \\ \tau_i \in Z^c_{\mathbf{e},\mathbf{q}}. \end{gathered} \tag{6.2}$$

Let $l \geq 1$. Then

$$n := l + m \geq 2.$$

Let $(\mathcal{A}, ^-)$ be the quantum torus determined by the vector $\mathbf{e}$ and the matrix $\mathbf{q}$. Let $\mathcal{G}$ be the tame generalized loop algebra of type C_n constructed in Section 4. (In Section 4, we denoted this algebra by $\mathcal{G}_0$.) Then $\mathcal{G}$ consists of all matrices $Y \in gl_{2n}(\mathcal{A})$ of the form

$$Y = \begin{bmatrix} A & B \\ C & -\bar{A}^t \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \bar{B}^t = B, \quad \bar{C}^t = C \text{ and} \\ tr(Y) \equiv 0 \bmod [\mathcal{A}, \mathcal{A}]. \end{array}$$

Put

$$F = \begin{bmatrix} x^{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{\tau_m} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 & I_\ell & 0 \\ 0 & -F^{-1} & 0 & 0 \\ -I_\ell & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{bmatrix}.$$

Then F and K are invertible matrices with

$$K^{-1} = \begin{bmatrix} 0 & 0 & -I_\ell & 0 \\ 0 & -F & 0 & 0 \\ I_\ell & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \end{bmatrix}.$$

By (2.8), we have

$$\bar{F}^t = -F \quad \text{and} \quad \bar{K}^t = -K. \tag{6.3}$$

Define

$$\begin{aligned} \psi_K &: \mathcal{G} \longrightarrow \mathcal{G}, \quad \text{by} \\ \psi_K(Y) &= -K^{-1}\bar{Y}^t K. \end{aligned} \tag{6.4}$$

Then one can easily see that ψ_K is a period 2 automorphism of $\mathcal{G}$. Denote by $\mathcal{G}_0$ the fixed point subalgebra of $\mathcal{G}$ with respect to ψ_k. Then

$$\mathcal{G}_0 = \{X := \psi_K(Y) + Y \mid Y \in \mathcal{G}\}.$$

Let

$$Y = \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & \bar{B}_2^t & B_4 \\ C_1 & C_2 & -\bar{A}_1^t & -\bar{A}_3^t \\ \bar{C}_2^t & C_4 & -\bar{A}_2^t & -\bar{A}_4^t \end{bmatrix} \in \mathcal{G} \quad \text{with} \quad \begin{array}{l} A_1, B_1, C_1 \in M_\ell(\mathcal{A}), \\ A_2, B_2, C_2 \in M_{l\times m}(\mathcal{A}), \\ A_4, B_4, C_4 \in M_m(\mathcal{A}), \\ \text{and} \quad A_3 \in M_{m\times l}(\mathcal{A}), \end{array}$$

where $\bar{B}_1^t = B_1$, $\bar{B}_4^t = B_4$, $\bar{C}_1^t = C_1$ and $\bar{C}_4^t = C_4$. Then

$$\psi_K(Y) + Y = \begin{bmatrix} 2A_1 & A_2 - B_2F^{-1} & 2B_1 & B_2 - A_2F \\ A_3 - F\bar{C}_2^t & A_4 - F\bar{A}_4^t F^{-1} & \bar{B}_2^t + F\bar{A}_2^t & B_4 + FC_4F \\ 2C_1 & C_2 + \bar{A}_3^t F^{-1} & -2\bar{A}_1^t & -\bar{A}_3^t - C_2F \\ \bar{C}_2^t - F^{-1}A_3 & C_4 + F^{-1}B_4F^{-1} & -\bar{A}_2^t - F^{-1}\bar{B}_2^t & -\bar{A}_4^t + F^{-1}A_4F \end{bmatrix}.$$

Therefore the general form of an element in $\mathcal{G}_0$ is

$$X = \begin{bmatrix} A & -\bar{D}^t & S & \bar{D}^t F \\ -FC & -\bar{B}^t & -FD & FPF \\ T & \bar{C}^t & -\bar{A}^t & -\bar{C}^t F \\ C & P & D & B \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \bar{S}^t = S, \quad \bar{T}^t = T, \quad \bar{P}^t = P, \\ F^{-1}\bar{B}^t F = -B, \quad \text{and} \\ tr(X) \equiv 0 \bmod [\mathcal{A}, \mathcal{A}]. \end{array} \tag{6.5}$$

where $A, S, T \in M_\ell(\mathcal{A})$, $C, D \in M_{m\times l}(\mathcal{A})$ and $B \in M_m(\mathcal{A})$. Let us denote a matrix of the form (6.5) by

$$X = X(A, B, C, D, P, S, T).$$

If $X(A, B, C, D, P, S, T)$ has the form (6.5) and we put

$$Y = \begin{bmatrix} A/2 & 0 & S/2 & \bar{D}^t F \\ 0 & -\bar{B}^t/2 & -FD & FPF \\ T/2 & \bar{C}^t & -\bar{A}^t/2 & 0 \\ C & 0 & 0 & B/2 \end{bmatrix}$$

then $Y \in \mathcal{G}$ and $X = Y + \psi(Y)$. Thus

$$\mathcal{G}_0 \text{ consists of all matrices } X(A, B, C, D, P, S, T) \text{ of the form (6.5).} \tag{6.6}$$

Let $(\cdot,\cdot)$ be the invariant nondegenerate symmetric bilinear form on $\mathcal{G}$, defined in Section 4. For $Y, Z \in \mathcal{G}$ we have

$$(\psi_K(Y), \psi_K(Z) = (-K^{-1}\bar{Y}^t K, -K^{-1}\bar{Z}^t K) = \epsilon(tr(K^{-1}\overline{ZY}^t K)) = \epsilon(tr(YZ)) = (Y, Z),$$

and so $(\cdot,\cdot)$ is invariant under ψ_K. Therefore by Lemma 2.15,

$$\text{the form } (\cdot,\cdot) \text{ restricted to } \mathcal{G}_0 \text{ is nondegenerate.} \tag{6.7}$$

Next, we put

$$\dot{\mathcal{H}}_0 = \{\sum_{i=1}^{l} a_i(e_{i,i} - e_{\ell+m+i,\ell+m+i}) \mid a_1, \dots, a_\ell \in \mathbb{C}\}.$$

Then,

$$\dot{\mathcal{H}}_0 \text{ is an abelian subalgebra of } \mathcal{G}_0. \tag{6.8}$$

We define $\epsilon_j \in \dot{\mathcal{H}}_0^*$, by

$$\epsilon_j(e_{ii} - e_{\ell+m+i,\ell+m+i}) = \delta_{ij}, \quad for \quad 1 \le i, j \le \ell.$$

We have

$$\mathcal{G}_0 = \sum_{\dot\alpha \in \dot{\mathcal{H}}_0^*} (\mathcal{G}_0)_{\dot\alpha} = (\mathcal{G}_0)_0 + \sum_{1 \le i \le \ell} \Big((\mathcal{G}_0)_{\epsilon_i} + (\mathcal{G}_0)_{-\epsilon_i} + (\mathcal{G}_0)_{2\epsilon_i} + (\mathcal{G}_0)_{-2\epsilon_i} \Big), \text{ if } l = 1 \text{ and}$$

$$\begin{aligned} \mathcal{G}_0 = \sum_{\dot\alpha \in \dot{\mathcal{H}}_0^*} (\mathcal{G}_0)_{\dot\alpha} = (\mathcal{G}_0)_0 + \sum_{1 \le i \ne j \le \ell} (\mathcal{G}_0)_{\epsilon_i - \epsilon_j} + \sum_{1 \le i < j \le \ell} \Big((\mathcal{G}_0)_{\epsilon_i + \epsilon_j} + (\mathcal{G}_0)_{-\epsilon_i - \epsilon_j} \Big) \\ + \sum_{1 \le i \le \ell} \Big((\mathcal{G}_0)_{\epsilon_i} + (\mathcal{G}_0)_{-\epsilon_i} + (\mathcal{G}_0)_{2\epsilon_i} + (\mathcal{G}_0)_{-2\epsilon_i} \Big), \text{ if } l \ge 2, \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} &(\mathcal{G}_0)_0 \text{ consists of all matrices of the form } X(A, B, 0, 0, P, 0, 0) \text{ in which} \\ &\qquad \text{the matrix } A \text{ is diagonal}, \end{aligned} \tag{6.10}$$

$$(\mathcal{G}_0)_{\epsilon_i - \epsilon_j} = \{ a e_{i,j} - \bar a e_{\ell+m+j, \ell+m+i} \mid a \in \mathcal{A} \}, \tag{6.11}$$

$$(\mathcal{G}_0)_{\epsilon_i + \epsilon_j} = \{ a e_{i, \ell+m+j} + \bar a e_{j, \ell+m+i} \mid a \in \mathcal{A} \}, \tag{6.12}$$

$$(\mathcal{G}_0)_{-\epsilon_i - \epsilon_j} = \{ a e_{\ell+m+i, j} + \bar a e_{\ell+m+j, i} \mid a \in \mathcal{A} \}, \tag{6.13}$$

$$\begin{aligned} (\mathcal{G}_0)_{\epsilon_i} = \{ \textstyle\sum_{j=1}^m (a_j e_{2\ell+m+j, \ell+m+i} - \bar a_j e_{i, \ell+j} \\ + \bar a_j x^{\tau_j} e_{i, 2\ell+m+j} - x^{\tau_j} a_j e_{\ell+j, \ell+m+i}) \mid a_j \in \mathcal{A} \}, \end{aligned} \tag{6.14}$$

$$\begin{aligned} (\mathcal{G}_0)_{-\epsilon_i} = \{ \textstyle\sum_{j=1}^m (a_j e_{2\ell+m+j, i} - \bar a_j e_{\ell+m+i, \ell+j} \\ - \bar a_j x^{\tau_j} e_{\ell+m+i, 2\ell+m+j} - x^{\tau_j} a_j e_{\ell+j, i}) \mid a_j \in \mathcal{A} \}, \end{aligned} \tag{6.15}$$

$$(\mathcal{G}_0)_{2\epsilon_i} = \{ h e_{i, \ell+m+i} \mid h \in \mathcal{A}_+ \} \text{ and} \tag{6.16}$$

$$(\mathcal{G}_0)_{-2\epsilon_i} = \{ h e_{\ell+m+i, i} \mid h \in \mathcal{A}_+ \}. \tag{6.17}$$

Let $\dot R = \{ \dot\alpha \in \dot{\mathcal{H}}_0^* \mid (\mathcal{G}_0)_{\dot\alpha} \ne 0 \}$. Then

$$\dot R = \begin{cases} \{0\} \cup \{ \pm\epsilon_i, \pm 2\epsilon_i \mid 1 \le i \le \ell \} & \text{if } l = 1 \\ \{0\} \cup \{ \pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \le i \ne j \le \ell \} & \text{if } l \ge 2. \end{cases} \tag{6.18}$$

Using (2.10) one can check easily that

$$(\epsilon_i, \epsilon_j) = \frac{1}{2}\delta_{ij} \quad \text{for } 1 \le i, j \le \ell. \tag{6.19}$$

Then

$$\dot R \text{ is an irreducible finite root system of type } BC_\ell \text{ in } \dot{\mathcal{V}} = \sum_{i=1}^{l} \mathbf{R}\epsilon_i. \tag{6.20}$$

To make $\mathcal{G}_0$ into a $\mathbf{Z}^\nu$-graded subalgebra of $gl_{2n}(\mathcal{A})$, we follow the same method as in Section 5. We define

$$deg(x^\sigma e_{pq}) = 2\sigma + \lambda_p - \lambda_q,$$

where

$$\lambda_1 = \cdots = \lambda_\ell = 0, \quad \lambda_{\ell+1} = -\tau_1, \ldots, \lambda_{\ell+m} = -\tau_m,$$
$$\lambda_{\ell+m+1} = \cdots = \lambda_{2\ell+m} = 0 \quad \text{and} \quad \lambda_{2\ell+m+1} = \tau_1, \ldots, \lambda_{2\ell+2m} = \tau_m.$$

This makes both the algebra $M_{2n}(\mathcal{A})$ and the Lie algebra $gl_{2n}(\mathcal{A})$ into $\mathbf{Z}^\nu$-graded algebras with $gl_{2n}(\mathcal{A})^\sigma = M_{2n}(\mathcal{A})^\sigma$, $\sigma \in \mathbf{Z}^\nu$. To show $\mathcal{G}_0$ is $\mathbf{Z}^\nu$-graded with respect to the above gradation, we check that $\mathcal{G}_0$, as a vector space over $\mathbf{C}$, is generated by homogenous elements with respect to the $\mathbf{Z}^\nu$-grading on $gl_{2n}(\mathcal{A})$. In particular, we need to show that $(\mathcal{G}_0)_0$ is generated by homogenous elements with respect to the $\mathbf{Z}^\nu$-grading on $gl_{2n}(\mathcal{A})$. This will be a consequence of the following Lemma. Recall that $(\mathcal{G}_0)_c$ is the subalgebra of $\mathcal{G}_0$ generated by spaces $(\mathcal{G}_0)_{\dot\alpha}$, $\dot\alpha \in \dot R^\times$.

Lemma 6.21 *We have*

$$(\mathcal{G}_0)_c = \{X(A, B, C, D, P, S, T) \mid P = -BF^{-1}\}.$$

Proof. Let us denote by M the right hand side of the equality in the statement. First we show that M is a subalgebra of $\mathcal{G}_0$. To see this let $X_i := X(A_i, B_i, C_i, D_i, P_i, S_i, T_i) \in \mathcal{G}_0$ for $i = 1, 2$, where $P_i = -B_i F^{-1}$. Then

$$[X_1, X_2] = X(*, B, *, *, P, *, *), \quad \text{where}$$
$$B = C_1\bar D_2^t F + P_1 F P_2 F - D_1 \bar C_2^t F + B_1 B_2 - C_2 \bar D_1^t F - P_2 F P_1 F + D_2 \bar C_1^t F - B_2 B_1, \text{ and}$$
$$P = -C_1 \bar D_2^t - P_1 \bar B_2^t + D_1 \bar C_2^t + B_1 P_2 + C_2 \bar D_1^t + P_2 \bar B_1^t - D_2 \bar C_1^t - B_2 P_1.$$

Since $P_i = -B_i F^{-1}$ and $F^{-1}\bar B_i^t F = -B_i$ for $i = 1, 2$, it is easy to see that $P = -BF^{-1}$. Thus $[X_1, X_2] \in M$. Next, we show that $(\mathcal{G}_0)_c \subseteq M$. By (6.11)-(6.17), M contains all spaces $(\mathcal{G}_0)_{\dot\alpha}$, $\dot\alpha \in \dot R^\times$. Thus $(\mathcal{G}_0)_c \subseteq M$.

Conversely, we show that $M \subseteq (\mathcal{G}_0)_c$. Let $X = X(A, B, C, D, -BF^{-1}, S, T) \in M$. We want to show $X \in (\mathcal{G}_0)_c$. By (6.11)-(6.17), $X(0, 0, C, D, 0, S, T) \in (\mathcal{G}_0)_c$. Subtracting this element from X, we can assume that $X = X(A, B, 0, 0, -BF^{-1}, 0, 0)$. By (6.11), each matrix of the form $X(A_1, 0, 0, 0, 0, 0, 0)$ in which A_1 has zero diagonal is in $(\mathcal{G}_0)_c$. So subtracting an element of this form from X we can assume that A is a diagonal matrix. Consider two matrices in $(\mathcal{G}_0)_c$ of the form $X_{C_1} := (0,0,be_{j,k},0,0,0,0)$ and $X_{D_1} := (0,0,0,-\bar{a}e_{i,k},0,0,0)$, $a,b \in \mathcal{A}$. Then

$$[X_{D_1}, X_{C_1}] = X(\delta_{ij}(-ax^{\tau_j}b - ax^{\tau_i}b)e_{kk}, \bar{a}\bar{b}x^{\tau_j}e_{ij} + bax^{\tau_i}e_{ji}, 0, 0, -\bar{a}\bar{b}e_{ij} - bae_{ji}, 0, 0) \quad (6.22)$$

is an element of $(\mathcal{G}_0)_c$. Take $b = 1$ in (6.22), to get the element

$$X(\delta_{ij}(-ax^{\tau_j} - ax^{\tau_i})e_{kk}, \bar{a}x^{\tau_j}e_{ij} + ax^{\tau_i}e_{ji}, 0, 0, -\bar{a}e_{ij} - ae_{ji}, 0, 0) \in (\mathcal{G}_0)_c. \quad (6.23)$$

From $F^{-1}\bar{B}^tF = -B$ it follows that $X(0,B,0,0,-BF^{-1},0,0)$ is a sum of elements of the form

$$X(0, \bar{a}x^{\tau_j}e_{ij} + ax^{\tau_i}e_{ji}, 0, 0, -\bar{a}e_{ij} - ae_{ji}, 0, 0), \quad a \in \mathcal{A}, \quad 1 \le i,j \le m.$$

Thus subtracting a sum of elements of the form (6.23) from X, we may assume that $X = X(A,0,0,0,0,0,0)$, A being diagonal. Now take $i = j = 1$ in (6.23) and replace a by $a(x^{\tau_1})^{-1}$, to get the element

$$X(-2ae_{kk}, (-(x^{\tau_1})^{-1}\bar{a}x^{\tau_1} + a)e_{11}, 0, 0, ((x^{\tau_1})^{-1}\bar{a} - a(x^{\tau_1})^{-1})e_{11}, 0, 0) \in (\mathcal{G}_0)_c. \quad (6.24)$$

Subtracting a sum of elements of this form from X, we may assume that

$$X = X(0, (-(x^{\tau_1})^{-1}\bar{a}x^{\tau_1} + a)e_{11}, 0, 0, ((x^{\tau_1})^{-1}\bar{a} - a(x^{\tau_1})^{-1})e_{11}, 0, 0).$$

Next, let

$$X_S = X(0,0,0,0,0,he_{11},0) \quad \text{and} \quad X_T = X(0,0,0,0,0,0,e_{11}), \quad h \in \mathcal{A}_+.$$

Then

$$[X_S, X_T] = (he_{11},0,0,0,0,0,0) \in (\mathcal{G}_0)_c, \quad h \in \mathcal{A}_+.$$

From this and (6.24), we get that

$$X(0, (-(x^{\tau_1})^{-1}hx^{\tau_1} + h)e_{11}, 0, 0, ((x^{\tau_1})^{-1}h - h(x^{\tau_1})^{-1})e_{11}, 0, 0)$$

is an element of $(\mathcal{G}_0)_c$ for all $h \in \mathcal{A}_+$. Subtracting an element of this form from X, we may assume that

$$X = X(0, ((x^{\tau_1})^{-1}sx^{\tau_1} + s)e_{11}, 0, 0, (-(x^{\tau_1})^{-1}s - s(x^{\tau_1})^{-1})e_{11}, 0, 0), \quad \text{where } s \in \mathcal{A}_-.$$

Since $tr(X) \equiv 0 \bmod [\mathcal{A},\mathcal{A}]$, we have

$$(x^{\tau_1})^{-1}sx^{\tau_1} + 2s + x^{\tau_1}s(x^{\tau_1})^{-1} \in [\mathcal{A},\mathcal{A}].$$

This implies that $s \in \mathcal{A}_- \cap [\mathcal{A},\mathcal{A}]$.

Now, take $i = j = k = 1$ in (6.22) and replace a by $a(x^{\tau_1})^{-1}$ to get the element

$$X(-2abe_{11},(-(x^{\tau_1})^{-1}\bar{a}\bar{b}x^{\tau_1} + ba)e_{11},0,0,((x^{\tau_1})^{-1}\bar{a}\bar{b} - ba(x^{\tau_1})^{-1})e_{11},0,0) \in (\mathcal{G}_0)_c. \quad (6.25)$$

If we replace b by 1 and a by ab in (6.25), we get the element

$$X(-2abe_{11},(-(x^{\tau_1})^{-1}\bar{b}\bar{a}x^{\tau_1} + ab)e_{11},0,0,((x^{\tau_1})^{-1}\bar{b}\bar{a} - ab(x^{\tau_1})^{-1})e_{11},0,0) \in (\mathcal{G}_0)_c. \quad (6.26)$$

The difference of the two elements in (6.25) and (6.26) is the element

$$X(0,((x^{\tau_1})^{-1}\overline{[a,b]}x^{\tau_1} - [a,b])e_{11},0,0,(-(x^{\tau_1})^{-1}\overline{[a,b]} + [a,b](x^{\tau_1})^{-1})e_{11},0,0) \in (\mathcal{G}_0)_c.$$

This implies

$$X(0,((x^{\tau_1})^{-1}([a,b] - \overline{[a,b]})x^{\tau_1} + ([a,b] - \overline{[a,b]}))e_{11},0,0,$$
$$(-(x^{\tau_1})^{-1}([a,b] - \overline{[a,b]}) - ([a,b] - \overline{[a,b]})(x^{\tau_1})^{-1})e_{11},0,0) \in (\mathcal{G}_0)_c.$$

But s is a sum of elements of the form $[a,b] - \overline{[a,b]}$ and so $X \in (\mathcal{G}_0)_c$. □

Corollary 6.27 *Elements in $\mathcal{G}_0$ of the form $X(A,B,0,0,0,0,0)$ with A and B diagonal matrices are in the $\mathbb{C}$-span of elements which have one of the following forms:*

$$(1)\ -2ax^{\tau_1}e_{kk} - 2x^{\tau_1}\bar{a}e_{\ell+m+k,\ell+m+k} + (\bar{a}x^{\tau_1} + ax^{\tau_1})e_{2\ell+m+i,2\ell+m+i} + (x^{\tau_1}a + x^{\tau_1}\bar{a})e_{\ell+i,\ell+i},$$
$$a \in \mathcal{A},\ 1 \le i \le m,\ 1 \le k \le \ell,$$

$$(2)\ -2ae_{kk} + 2\bar{a}e_{\ell+m+k,\ell+m+k} + (-(x^{\tau_1})^{-1}\bar{a}x^{\tau_1} + a)\,e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}a(x^{\tau_1})^{-1} - \bar{a})e_{\ell+1,\ell+1},$$
$$a \in \mathcal{A},\ 1 \le k \le \ell.$$

$$(3)\,(-(x^{\tau_1})^{-1}hx^{\tau_1} + h)e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}h(x^{\tau_1})^{-1} - h)e_{\ell+1,\ell+1},\quad h \in \mathcal{A}_+,$$

$$(4)\,((x^{\tau_1})^{-1}sx^{\tau_1} + s)e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}s(x^{\tau_1})^{-1} + s)e_{\ell+1,\ell+1},\quad s \in \mathcal{A}_- \cap [\mathcal{A},\mathcal{A}].$$

Proof. By (6.23), (6.24) and (6.25) of the proof of Lemma 6.21, we have that all elements of the form (1)-(4), in the statement have trace zero modulo $[\mathcal{A},\mathcal{A}]$ and so are in $\mathcal{G}_0$. Now let $X = X(A, B, 0, 0, 0, 0, 0)$ is in $\mathcal{G}_0$ with A and B diagonal matrices. Since $F^{-1}\bar{B}^t F = -B$, B is a sum of elements of the form

$$(\bar{a}x^{\tau_i} + ax^{\tau_i})e_{2\ell+m+i,2\ell+m+i} + (x^{\tau_i}a + x^{\tau_i}\bar{a})e_{\ell+i,\ell+i}, \quad a \in \mathcal{A}, \quad 1 \leq i \leq m.$$

Thus subtracting a sum of elements of the form (1) from X we can assume that $B = 0$, that is $X = X(A, 0, 0, 0, 0, 0, 0)$, A a diagonal matrix. Next subtracting a sum of elements of the form (2) from X, we can assume that X is of the form

$$X = (-(x^{\tau_1})^{-1}\bar{a}x^{\tau_1} + a)e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}a(x^{\tau_1})^{-1} - \bar{a})e_{\ell+1,\ell+1}, \quad a \in \mathcal{A}.$$

Since $tr(X) \equiv 0 \mod [\mathcal{A}, \mathcal{A}]$, it follows that X is a sum of elements of the form (3) and (4).

□

From (6.10) and Corollary 6.27 we have the following result.

Corollary 6.28 *$(\mathcal{G}_0)_0$, as a complex vector space, is generated by elements which have one of the following forms:*

(1) $$-2x^{\sigma}x^{\tau_i}e_{kk} - 2x^{\tau_i}\bar{x^{\sigma}}e_{\ell+m+k,\ell+m+k} + (\bar{x^{\sigma}}x^{\tau_i} + x^{\sigma}x^{\tau_i})e_{2\ell+m+i,2\ell+m+i} + (x^{\tau_i}x^{\sigma} + x^{\tau_i}\bar{x^{\sigma}})e_{\ell+i,\ell+i},$$
$$\sigma \in \mathbf{Z}^{\nu}, \ 1 \leq k \leq \ell, \ 1 \leq i \leq m,$$

(2) $$-2x^{\sigma}e_{kk} + 2\bar{x^{\sigma}}e_{\ell+m+k,\ell+m+k} + \left(-(x^{\tau_1})^{-1}\bar{x^{\sigma}}x^{\tau_1} + x^{\sigma}\right)e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}x^{\sigma}(x^{\tau_1})^{-1} - \bar{x^{\sigma}})e_{\ell+1,\ell+1},$$
$$\sigma \in \mathbf{Z}^{\nu}, \ 1 \leq k \leq \ell,$$

(3) $(-(x^{\tau_1})^{-1}x^{\sigma}x^{\tau_1} + x^{\sigma})e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}x^{\sigma}(x^{\tau_1})^{-1} - x^{\sigma})e_{\ell+1,\ell+1}$, $\sigma \in Z_{e,q}$,

(4) $((x^{\tau_1})^{-1}sx^{\tau_1} + s)e_{2\ell+m+1,2\ell+m+1} + (x^{\tau_1}s(x^{\tau_1})^{-1} + s)e_{\ell+1,\ell+1}$,
$$s = [x^{\sigma}, x^{\eta}] - \overline{[x^{\sigma}, x^{\eta}]}, \quad \sigma, \eta \in \mathbf{Z}^{\nu},$$

(5) $x^{\sigma}x^{\tau_j}e_{2\ell+m+i,2\ell+m+j} + \bar{x^{\sigma}}x^{\tau_i}e_{2\ell+m+j,2\ell+m+i}$
$$+x^{\tau_j}\bar{x^{\sigma}}e_{\ell+j,\ell+i} + x^{\tau_i}x^{\sigma}e_{\ell+i,\ell+j}, \ \sigma \in \mathbf{Z}^{\nu}, \ 1 \leq i < j \leq m,$$

(6) $x^{\sigma}e_{2\ell+m+i,\ell+j} + \bar{x^{\sigma}}e_{2\ell+m+j,\ell+i} + x^{\tau_i}x^{\sigma}x^{\tau_j}e_{\ell+i,2\ell+m+j}$
$$+x^{\tau_j}\bar{x^{\sigma}}x^{\tau_i}e_{\ell+j,2\ell+m+i}, \ \sigma \in \mathbf{Z}^{\nu}, \ 1 \leq i \leq j \leq m.$$

□

In Corollary 6.40, we have that

$$\begin{gathered}\text{each element of the form (1) has degree } 2\sigma+2\tau_i,\ \sigma\in\mathbf{Z}^\nu,\ 1\le i\le m,\\ \text{each element of the form (2) has degree } 2\sigma,\ \sigma\in\mathbf{Z}^\nu,\\ \text{each element of the form (3) has degree } 2\sigma,\ \sigma\in Z_{\mathbf{e},\mathbf{q}},\\ \text{each element of the form (4) has degree } 2\sigma, \text{ for some nonzero } \sigma\in\mathbf{Z}^\nu,\\ \text{each element of the form (5) has degree } 2\sigma+\tau_i+\tau_j,\ \sigma\in\mathbf{Z}^\nu,\ 1\le i<j\le m,\\ \text{each element of the form (6) has degree } 2\sigma+\tau_i+\tau_j,\ \sigma\in\mathbf{Z}^\nu,\ 1\le i\le j\le m.\end{gathered}\tag{6.29}$$

From (6.11), (6.12) and (6.13) we see that

$$(\mathcal{G}_0)_{\epsilon_i-\epsilon_j},\ (\mathcal{G}_0)_{\epsilon_i+\epsilon_j} \text{ and } (\mathcal{G}_0)_{-\epsilon_i-\epsilon_j} \text{ are generated by elements of degree } 2\sigma,\ \sigma\in\mathbf{Z}^\nu.\tag{6.30}$$

From (6.14) and (6.15), we see that

$$\begin{gathered}(\mathcal{G}_0)_{\epsilon_i} \text{ and } (\mathcal{G}_0)_{-\epsilon_i} \text{ are generated by elements of degree}\\ 2\sigma+\tau_j,\ \sigma\in\mathbf{Z}^\nu,\ 1\le j\le m.\end{gathered}\tag{6.31}$$

Finally, from (6.16), (6.17) and (2.8), we see that

$$\begin{gathered}(\mathcal{G}_0)_{2\epsilon_i} \text{ and } (\mathcal{G}_0)_{-2\epsilon_i} \text{ are generated by elements of the forms}\\ x^\sigma e_{i,\ell+m+i} \text{ and } x^\sigma e_{\ell+m+i,i},\ \sigma\in Z_{\mathbf{e},\mathbf{q}},\\ \text{having degree } 2\sigma.\end{gathered}\tag{6.32}$$

By (6.18), Corollary 6.28 and (6.29)-(6.32), we have

$$(\mathcal{G}_0)_{\dot\alpha}=\sum_{\sigma\in\mathbf{Z}^\nu}\mathcal{G}_0^\sigma\cap(\mathcal{G}_0)_{\dot\alpha}\quad\text{for all } \dot\alpha\in\dot R.\tag{6.33}$$

In particular, $\mathcal{G}_0$ has a basis consisting of homogeneous elements with respect to the $\mathbf{Z}^\nu$-grading on $gl_{2n}(\mathcal{A})$. Hence,

$$\mathcal{G}_0 \text{ is } \mathbf{Z}^\nu\text{-graded with } \mathcal{G}_0=\bigoplus\sum_{\sigma\in\mathbf{Z}^\nu}\mathcal{G}_0^\sigma, \text{ where } \mathcal{G}_0^\sigma=\mathcal{G}_0\cap gl_{2n}(\mathcal{A})^\sigma.\tag{6.34}$$

Lemma 6.35 *The triple $(\mathcal{G}_0,(\cdot,\cdot),\dot{\mathcal{H}}_0)$ satisfies conditions (C1) through (C11) and so is a generalized loop algebra.*

Proof. $\mathcal{G}_0$ satisfies (C1), by (6.7). By (6.9), the triple $(\mathcal{G}_0,(\cdot,\cdot),\dot{\mathcal{H}}_0)$ satisfies (C2). (C3), (C4), (C5) and (C6) follows from (6.19), (6.20), (6.34) and (6.33) respectively. For (C7), we first see from Corollary 6.28 that $\dot{\mathcal{H}}_0 \subseteq \mathcal{G}_0 \cap \mathcal{G}_0^0$. Then the equality follows from Corollary 6.28, (6.29) and (6.2). By (6.32), $2\mathbf{Z}^\nu \subseteq \{\sigma \in \mathbf{Z}^\nu \mid \mathcal{G}^\sigma \neq \{0\}\}$, and so (C8) holds. For (C9), it is enough to show that $(M_{2n}(\mathcal{A})^\sigma, M_{2n}(\mathcal{A})^\tau) = \{0\}$ if $\sigma, \tau \in \mathbf{Z}^\nu$ with $\sigma + \tau \neq 0$. But for $1 \leq i, j, p, q \leq 2n$ and $\sigma, \tau \in \mathbf{Z}^\nu$, we have

$$(x^\sigma e_{ij}, x^\tau e_{pq}) = \epsilon(tr(x^\sigma e_{ij} x^\tau e_{pq})) = \pm\delta_{jp}\delta_{iq}\delta_{\sigma,-\tau}$$

which is zero if $2\sigma + \lambda_i - \lambda_j + 2\tau + \lambda_p - \lambda_q \neq 0$. Thus (C9) holds.

We now consider (C10). We have $X = BC_\ell$, by (6.20). By (6.30) and (6.31), we have

$$\mathcal{G}_0^0 \cap (\mathcal{G}_0)_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot R_{lg}, \quad \text{and} \quad \mathcal{G}_0^{\tau_1} \cap (\mathcal{G}_0)_{\dot\alpha} \neq \{0\} \text{ for } \dot\alpha \in \dot R_{sh}.$$

Thus (C10) hols with $\tau = \tau_1$.

For (C11), let $\sigma \in R^0$. By (1.15) and (6.29), we have

$$R^0 = \{2\mathbf{Z}^\nu + \tau_i + \tau_j \mid 1 \leq i, j \leq m\}.$$

Therefore $\sigma = 2\xi + \tau_i + \tau_j$ for some $\xi \in \mathbf{Z}^\nu$, $1 \leq i, j \leq m$. Let $\eta = -\tau_i$. Then $\eta + \sigma = 2\xi + \tau_j$. By (6.31), we have

$$(\mathcal{G}_0)_{\epsilon_1} \cap \mathcal{G}_0^\eta \neq \{0\} \quad \text{and} \quad (\mathcal{G}_0)_{\epsilon_1} \cap \mathcal{G}_0^{\eta+\sigma} \neq \{0\}.$$

Thus (C11) holds. This finishes the proof of Lemma. □

Proposition 6.36 *Suppose $l \geq 1$, $m \geq 1$ and $\tau_1, \ldots, \tau_m$ satisfy (6.2). Let $(\mathcal{G}_0, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ be the triple defined above. Then the triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ constructed in Section 1 (with $\mathcal{G}_0$ and $\dot{\mathcal{H}}_0$ in place of $\mathcal{G}$ and $\dot{\mathcal{H}}$ respectively), is an EALA of type BC_ℓ and nullity ν. Moreover, the root system of $\mathcal{L}$ is*

$$R \cong \begin{cases} R(BC_1, S, E) & \text{if } l = 1 \\ R(BC_\ell, S, L, E) & \text{if } l \geq 2, \end{cases}$$

where $S = \cup_{j=1}^m (2\mathbf{Z}^\nu + \tau_j + \tau_1)$, $L = 2\mathbf{Z}^\nu$ and $E = 2(Z_{\mathbf{e},\mathbf{q}} + \tau_1)$.

Proof. By Lemma 6.35 and Proposition 1.21, $\mathcal{L}$ is an EALA of nullity ν. So it remains to prove the statement regarding the root system. By (1.14), (6.11)-(6.17) and (6.29)-(6.32), we have

$$R^\times = \begin{cases} \left(\cup_{i=1}^m(\dot R_{sh} + 2\mathbf{Z}^\nu + \tau_i)\right) \cup (\dot R_{ex} + 2Z_{\mathbf{e},\mathbf{q}}) & \text{if } l = 1 \\ \left(\cup_{i=1}^m(\dot R_{sh} + 2\mathbf{Z}^\nu + \tau_i)\right) \cup (\dot R_{lg} + 2\mathbf{Z}^\nu) \cup (\dot R_{ex} + 2Z_{\mathbf{e},\mathbf{q}}) & \text{if } l \geq 2 \end{cases}$$

and

$$R^0 = \{2\mathbf{Z}^\nu + \tau_i + \tau_j \mid 1 \leq i, j \leq m\}.$$

Let $\tau = \tau_1$. Then we can write R in terms of $\dot{R}_\tau$ (see Definition 1.18) as

$$R = R^0 \cup R^\times = \begin{cases} (S+S) \cup \Big((\dot{R}_\tau)_{sh} + S\Big) \cup \Big((\dot{R}_\tau)_{ex} + E\Big) & \text{if } l = 1 \\ (S+S) \cup \Big((\dot{R}_\tau)_{sh} + S\Big) \cup \Big((\dot{R}_\tau)_{lg} + L\Big) \cup \Big((\dot{R}_\tau)_{ex} + E\Big) & \text{if } l \geq 2 \end{cases}$$

where S, L and E are as in the statement. This finishes the proof of Proposition. □

Remark 6.37 *We claim that the root systems of type BC appearing in Proposition 6.36 are the same as those which we get in Theorem 3.8 (up to isomorphism). To see this let (S, L, E) be a triple of the form appearing in Theorem 3.8. Then*

$$S = \cup_{i=1}^m (\tau_i + 2\mathbf{Z}^\nu), \quad L = 2\mathbf{Z}^\nu, \quad E = 2Z^c_{\mathbf{e},\mathbf{q}},$$

where $\mathbf{e}$ and $\mathbf{q}$ satisfy (2.1) and $\tau_1, \ldots, \tau_m$ are elements of $\mathbf{Z}^\nu$ satisfying (3.1). Moreover either $\mathbf{e} \neq 1_\nu$ or $\mathbf{q} \neq 1_{\nu\times\nu}$. Take

$$0 \neq \tau_1' \in Z^c_{\mathbf{e},\mathbf{q}}.$$

By Lemma 2.6, there exists a quadratic form

$$Q((n_1, \ldots, n_\nu)) = \sum_{i\in I} n_i^2 + \sum_{(i,j)\in J} n_i n_j,$$

where $I \subseteq \{1, \ldots, \nu\}$ and $J \subseteq \{(i,j) \mid 1 \leq i < j \leq \nu\}$, such that

$$Z(Q) = Z^c_{\mathbf{e},\mathbf{q}} + \tau_1'.$$

Let $\mathbf{e}' = (\mathbf{e}_1', \ldots, \mathbf{e}_\nu')$ and $\mathbf{q}' = (\mathbf{q}_{ij}')_{1\leq i,j\leq\nu}$ where

$$\mathbf{e}_i' = -1 \text{ if } i \in I \quad \text{and} \quad \mathbf{e}_i' = 1 \text{ if } i \notin I.$$
$$\mathbf{q}_{ij}' = -1 \text{ if } (i,j) \in J \quad \text{and} \quad \mathbf{q}_{ij}' = 1 \text{ if } (i,j) \notin J.$$

Then $\mathbf{e}'$ and $\mathbf{q}'$ satisfy (2.1) and $Q = Q_{\mathbf{e}',\mathbf{q}'}$. Thus

$$Z_{\mathbf{e}',\mathbf{q}'} = Z(Q) = Z^c_{\mathbf{e},\mathbf{q}} + \tau_1'.$$

Put

$$\tau_i' = \tau_1' + \tau_i \quad \text{for } i = 2, \ldots, m.$$

Then $\tau_1', \ldots, \tau_m'$ satisfy (3.1). Now corresponding to the vector $\mathbf{e}'$, matrix $\mathbf{q}'$ and elements $\tau_1', \ldots, \tau_m'$, the triple (S', L', E') appearing in Proposition 6.36 has the form

$$S' = \cup_{i=1}^{m}(2\mathbf{Z}^{\nu} + \tau_1' + \tau_i'), \quad L' = 2\mathbf{Z}^{\nu}, \quad E' = 2(Z_{\mathbf{e}',\mathbf{q}'} + \tau_1').$$

Therefore

$$S' = \cup_{i=1}^{m}(2\mathbf{Z}^{\nu} + \tau_i) = S, \quad L' = L, \quad \text{and} \quad E' = 2(Z_{\mathbf{e}',\mathbf{q}'} + \tau_1') = 2Z_{\mathbf{e},\mathbf{q}}^{c}.$$

Thus $(S, L, E) = (S', L', E')$. Therefore the two root systems appearing in Theorem 3.8 and Proposition 6.36 are iomorphic, by Theorem I.1.29. □

In what follows let $\mathcal{L}$ be the EALA constructed from $\mathcal{G}_0$ as in Proposition 6.36. By Corollary 1.30,

$$\mathcal{L}_c = (\mathcal{G}_0)_c + \mathcal{C}. \tag{6.38}$$

Recall from (1.32) and (I.1.42) that

$$(\mathcal{G}_0)_c^{\perp} = \{X \in \mathcal{G}_0 \mid [X, (\mathcal{G}_0)_c] = \{0\}\} \quad \text{and} \quad \mathcal{L}_c^{\perp} = \{X \in \mathcal{L} \mid [X, \mathcal{L}_c] = \{0\}\}.$$

Lemma 6.39

$$(\mathcal{G}_0)_c^{\perp} = \{X(aI_{\ell}, B, 0, 0, P, 0, 0) \in \mathcal{G}_0 \mid -PF + B = aI_m,\ a \in \mathit{Cent}(\mathcal{A}) \cap \mathcal{A}_-\}.$$

Proof. First, we show that $(\mathcal{G}_0)_c^{\perp}$ is a subset of the right hand side of the equality in the statement. Let $X = X(A, B, C, D, P, S, T) \in (\mathcal{G}_0)_c^{\perp}$. By Lemma 6.21,

$$X_{A_1} = X(A_1, 0, 0, 0, 0, 0, 0) \in (\mathcal{G}_0)_c \text{ for all } A_1 \in M_{\ell}(\mathcal{A}),\ A \text{ having zero diagonal.}$$

Then

$$0 = [X, X_{A_1}] = X(AA_1 - A_1A, 0, CA_1, D\bar{A}_1^t, -S\bar{A}_1^t - A_1S, TA_1 + \bar{A}_1^tT) \text{ for all } A_1 \in M_{\ell}(\mathcal{A}).$$

From this, we get $C = 0$, $D = 0$, $S = 0$, $T = 0$ and $A = aI_{\ell}$ for some $a \in \text{Cent}(\mathcal{A})$. Thus $X = X(A, B, 0, 0, P, 0, 0)$ with $A = aI_{\ell}$, $a \in \text{Cent}(\mathcal{A})$. Next, we have

$$X_{S_1} = X(0, 0, 0, 0, 0, S_1, 0) \in (\mathcal{G}_0)_c \text{ for all } S_1 \in M_{\ell}(\mathcal{A}) \text{ with } \bar{S}_1^t = S_1.$$

Thus

$$0 = [X, X_{S_1}] = X(0, 0, 0, 0, 0, AS_1 + S_1\bar{A}^t, 0).$$

Take S_1 to be the identity matrix to get $A + \bar{A}^t = 0$. Therefore $a \in \mathcal{A}_-$. Next, we have

$$X_{D_1} = X(0, 0, 0, D_1, 0, 0, 0) \in (\mathcal{G}_0)_c \text{ for all matrices } D_1 \in M_{m\times l}.$$

Thus

$$0 = [X, X_{D_1}] = X(0, 0, 0, -PFD_1 + BD_1 + D_1\bar{A}^t, 0, 0, 0), \quad D_1 \in M_{m\times \ell}.$$

Since $a \in \mathrm{Cent}(\mathcal{A}) \cap \mathcal{A}_-$, the above equality gives

$$-PF + B = aI_m.$$

Thus X is in the right hand side of the equality in the statement.

Conversely, let $X(A, B, 0, 0, P, 0, 0) \in \mathcal{G}_0$ where $A = aI_\ell$, $-PF + B = aI_m$ and $a \in \mathrm{Cent}(\mathcal{A}) \cap \mathcal{A}_-$. Then, one can easily check that $[X, Y] = 0$ for all $Y \in (\mathcal{G}_0)_c$. This completes the proof. □

Corollary 6.40 $(\mathcal{G}_0)_c \cap (\mathcal{G}_0)_c^\perp = \{0\}$.

Proof. Let $X \in (\mathcal{G}_0)_c \cap (\mathcal{G}_0)_c^\perp$. By Lemmas (6.21) and (6.39), we have

$$X = X(aI_m, B, 0, 0, P, 0, 0)$$

where $B + PF = 0$ and $-PF + B = aI_m$, $a \in \mathrm{Cent}(\mathcal{A}) \cap \mathcal{A}_-$. From this we get $B = (1/2)aI_m$, $a \in \mathrm{Cent}(\mathcal{A}) \cap \mathcal{A}_-$. Since $\bar{a} = -a$, $\mathrm{tr}(X) = (2\ell + m)a$. Since $\mathrm{tr}(X) = 0 \bmod [\mathcal{A}, \mathcal{A}]$, we get $a \in [\mathcal{A}, \mathcal{A}]$. By [BGK,2.44(iii)], $\mathrm{Cent}(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}] = \{0\}$, and so $a = 0$. Hence $X = 0$. □

Corollary 6.41 *The EALA $\mathcal{L}$ might not be tame. In particular if $\mathcal{A}_- \cap \mathrm{Cent}(\mathcal{A}) \neq \{0\}$, then $\mathcal{L}$ is not tame.*

Proof. $\mathcal{L}$ is tame if and only if $\mathcal{L}_c^\perp \subseteq \mathcal{L}_c$. By Lemma 1.33, we have $(\mathcal{G}_0)_c^\perp \subseteq \mathcal{L}_c^\perp$. By Corollary 6.40, we have $(\mathcal{G}_0)_c^\perp \cap \mathcal{L}_c = \{0\}$ (since $\mathcal{L}_c = (\mathcal{G}_0)_c \oplus \mathcal{C}$). We are done if we show that $(\mathcal{G}_0)_c^\perp \neq \{0\}$. For this, consider $0 \neq a \in \mathrm{Cent}(\mathcal{A}) \cap \mathcal{A}_-$ and let

$$B = l\left(-(x^{\tau_1})^{-1}\bar{a}x^{\tau_1} + a\right)e_{2\ell+m+1,2\ell+m+1} + l\left(x^{\tau_1}a(x^{\tau_1})^{-1} - \bar{a}\right)e_{\ell+1,\ell+1}.$$

Then the element

$$X(2aI_\ell, B, 0, 0, 0, 0, 0)$$

is a sum of elements of the form (2) in Corollary 6.27, and so is an element of $\mathcal{G}_0$. Now

$$X = X(2aI_\ell, B, 0, 0, BF^{-1} - aF^{-1}, 0, 0)$$

is a nonzero element of $\mathcal{G}_0$ which is in $(\mathcal{G}_0)_c^\perp$, by Lemma 6.39. □

Lemma 6.42 $((\mathcal{G}_0)_c, (\cdot,\cdot), \dot{\mathcal{H}}_0)$ *is a tame generalized loop algebra.*

Proof. By Lemma 1.34, we only need to show that $\dot{\mathcal{H}}_0 \subseteq (\mathcal{G}_0)_c$ and that the form $(\cdot,\cdot)$ restricted to $(\mathcal{G}_0)_c$ is nondegenerate. But these follow respectively from Lemmas 6.21 and 6.40. □

From Lemma 6.41 and Proposition 1.21, we get the following result.

Proposition 6.43 *The Lie algebra* $(\mathcal{G}_0)_c \oplus \mathcal{C} \oplus \mathcal{D}$ *constructed in Section 1 (with* $(\mathcal{G}_0)_c$ *in place of* $\mathcal{G}$*) is a tame EALA of type* BC_ℓ *of nullity* ν. *Moreover, the root system* R *of* $\mathcal{L}$ *is isomorphic to*

$$R \cong \begin{cases} R(BC_1, S, E) & \text{if } l = 1 \\ R(BC_\ell, S, L, E) & \text{if } l \geq 2, \end{cases}$$

where $S = \cup_{j=1}^m (2\mathbf{Z}^\nu + \tau_j + \tau_1)$, $L = 2\mathbf{Z}^\nu$ *and* $E = 2(Z_{\mathbf{e},\mathbf{q}} + \tau_1)$.

Chapter 3

Extended Affine Weyl Groups

Introduction

The main object of study in this chapter is the Weyl group of an extended affine root system which we call an extended affine Weyl group (EAWG for short). The definition of an EAWG is a natural generalization of the definition of an affine Kac-Moody Weyl group (see Definition 2.15). Finite and affine Weyl groups are examples of EAWG's. Moreover, toroidal Weyl groups, the Weyl groups of troidal root systems, are also EAWG's.

There is no need to emphasize the importance of Weyl groups in the theory of Lie algebras. In particular, affine Weyl groups are among the most important objects in the theory of affine Kac-Moody algebras. It is natural to expect a similar role for EAWG's in the theory of EALA's (see [Sa1], [BGK], [Kr] and [AABGP] for some applications of EAWG in understanding of theory of EALA's and EARS's).

In 1985 K. Saito [Sa1], introduced axioms for EARS's (the root systems which Saito considered contains only nonisotropic roots). He studied these root systems and their Weyl groups and classified (marked) EARS's of nullity ν. For Saito the Weyl group of an EARS is a subgroup of automorphisms of the real span of roots. Even though that time there was no specific description of the structure of EARS's of nullity > 2, he was able to (using Eichler-Siegel map) describe the Weyl group as a semidirect product of a finite Weyl group and an abelian normal subgroup. He also obtained some results regarding a central extension of such Weyl groups. In 1992, Moody and Shi [M-S] studied toroidal Weyl groups, the Weyl groups of toroidal root systems. Their work covers all EAWG's where the underlying root system R is simply laced with rank > 1. They also considered some special cases when R is of type A_1. Regarding the structural theory of EAWG's the basic achievements in [M-S] is the unique expression they give for a typical element of the Weyl group in terms of some naturally arisen transformations (Eichler-Siegel transformations), where the knowledge of the structure of toroidal roots systems plays an important role in this achievement. Based on the knowledge of the structure of EARS's investigated in [AABGP, Chapters II, III] (see Construction I.1.24 and Theorems I.1.28 and I.1.29), our approach in the study of EAWG's will be similar to that of [M-S]. In fact the current chapter, in part, is a generalization of the [M-S]'s paper to all EAWG's when the root system R is reduced, i.e. R has one of the types A, D, E, B, C, F and G.

Section 1 gives the basic definitions and results about semilattices and the structure of EARS's. The notion of index for a semilattice, given in Definition 1.19, plays an important role in section 4 (see also [A]). In section 2, the general setup for the study of EAWG's is provided. The notion of duality for finite and affine root systems is generalized to the class of EARS's. (Saito [Sa1] discussed this notion only for the set of nonisotropic roots.)

Section 3 deals with the structure of an EAWG $\mathcal{W}$. It is shown that $\mathcal{W}$ is a semidirect product of a finite Weyl group $\dot{\mathcal{W}}$ and a characteristic subgroup H of $\mathcal{W}$, where H is a 2-step nilpotent abelian group with a center which is a free abelian group of rank $\nu(\nu-1)/2$, ν being the nullity of the EARS. Corollary 3.25 and Proposition 3.30 provide a unique expression of elements in $\mathcal{W}$ in terms of some naturally arisen transformations (Eichler-Siegel transformations). We regard this as the basic acheivement in Section 3.

In section 4, one of the characterizations of a basis for a finite or affine root system is considered for the class of EARS's. Namely, for an EARS R of type X, there exists a subset $\Pi(X)$ of nonisotropic roots so that all the nonisotropic roots can be recovered by the action on $\Pi(X)$ of the subgroup $\mathcal{W}_{\Pi(X)}$ of $\mathcal{W}$ generated by reflections r_α, $\alpha \in \Pi(X)$. Moreover, $\Pi(X)$ has the least cardinality $ind(R)+\ell+\nu$ with this property, where $ind(R)$, is an isomorphic invariant of R, ℓ is the rank of the finite root system attached to R and ν is the nullity of R. In the finite or affine case, when $\nu = 0$ or 1, $\Pi(X)$, $\mathcal{W}_{\Pi(X)}$ and $\text{ind}(R)+\ell+\nu$ are just a set of simple roots, the Weyl group and the rank of R, respectively.

Section 5 is devoted to the study of Weyl groups of EARS's of index zero. The main result of this section shows that an EARS of index zero has a presentation by conjugation (Theorem 5.17).

The final section, Section 6, is a generalization of Section 2 of [M-S], showing that the Weyl group of an EARS of index zero is the homomorphic image of some Weyl groups of indefinite type where the homomorphism and its kernel are given explicitly.

1 Basics

In this section we discuss the basic information which we need to start the study of the Weyl group of an extended affine root system (EARS).

Let R be an EARS in a finite dimensional vector space $\mathcal{V}$, as in Definition I.1.14. Then $\mathcal{V}$ is equipped with a semidefinite symmetric bilinear form $(\cdot,\cdot)$ and R satisfies axioms (R1)-(R8) of Definition I.1.14. We have

$$R = R^{\times} \uplus R^0, \quad \text{where}$$

$$R^{\times} = \{\alpha \in R \mid (\alpha,\alpha) \neq 0\}, \text{ the set of nonisotropic roots of } R \text{ and}$$
$$R^0 = \{\alpha \in R \mid (\alpha,\alpha) = 0\}, \text{ the set of isotropic roots of } R.$$

Recall from Chapter 1 that if $\mathcal{V}^0$ is the radical of $(\cdot,\cdot)$, $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and $^{-} : \mathcal{V} \to \bar{\mathcal{V}}$ is the canonical map and we define

$$(\bar{\alpha},\bar{\beta}) = (\alpha,\beta), \quad \text{for } \alpha,\beta \in \mathcal{V},$$

then

$$\begin{gathered}(\cdot,\cdot) \text{ is a symmetric positive definite bilinear form on } \bar{\mathcal{V}} \text{ and}\\ \bar{R} = \{\bar{\alpha} \mid \alpha \in R\} \text{ is a finite irreducible root system in } \bar{\mathcal{V}}.\end{gathered} \tag{1.1}$$

Let $\mathcal{V}^0$ be ν-dimensional and $\bar{R}$ have rank ℓ. Then by Definition I.1.16, R has nullity ν, rank ℓ and type X, where X denotes the type of finite root system $\bar{R}$. Also R is reduced if $\bar{R}$ is.

Throughout this work we assume that

$$\begin{gathered}R \text{ is a reduced EARS. That is, } R \text{ has one of the types}\\ X = A_\ell(l \geq 1),\ D_\ell(l \geq 4),\ E_6,\ E_7,\ E_8,\ B_\ell(l \geq 2),\ C_\ell(l \geq 3),\ F_4 \text{ or } G_2.\\ \text{Thus we consider all types except for type } BC_\ell(l \geq 1).\end{gathered}$$

As in (I.1.17), we fix a basis $\bar{\Pi} = \{\alpha_1,\ldots,\alpha_\ell\}$ of $\bar{R}$ and choose $\dot{\alpha}_i$ in R so that $\bar{\dot{\alpha}}_i = \alpha_i$, we let $\dot{\mathcal{V}}$ be the real span of $\dot{\alpha}_1,\ldots,\dot{\alpha}_\ell$ and

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \dot{\alpha} + \delta \in R \text{ for some } \delta \in \mathcal{V}^0\}. \tag{1.2}$$

Then $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and by (I.1.19), $\dot{R}$ is a finite root system in $\dot{\mathcal{V}}$ isomorphic to $\bar{R}$. Moreover,

$$\dot{R} \subseteq R, \tag{1.3}$$

since R is reduced. As in Chapter II, we assume that

$$\text{if X is simply laced then every root is a short root.} \tag{1.4}$$

As usual we decompose $\dot{R}^\times := \dot{R} \setminus \{0\}$ as

$$\dot{R}^\times = \dot{R}_{sh} \cup \dot{R}_{lg}, \tag{1.5}$$

where $\dot{R}_{sh}$ is the set of short roots and $\dot{R}_{lg}$ is the set of long roots (which might be empty) of $\dot{R}^\times$. Let

$$\begin{aligned} S &= \{\delta \in \mathcal{V}^0 \mid \alpha + \delta \in R, \text{ for some } \alpha \in \dot{R}_{sh}\} \quad \text{and} \\ L &= \{\delta \in \mathcal{V}^0 \mid \alpha + \delta \in R, \text{ for some } \alpha \in \dot{R}_{lg}\}, \quad (\text{if } \dot{R}_{lg} \neq \emptyset). \end{aligned} \tag{1.6}$$

As in (I.1.21),

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L),$$

in which S and L are semilattices (see Definition I.1.23) where we interpret the term $\dot{R}_{lg} + L$ as an empty set if $\dot{R}_{lg}$ is empty. More precisely, from Construction I.1.24 and Theorem I.1.28, we have

$$\begin{gathered} R = (S + S) \cup (\dot{R} + S) \quad \text{if } X \text{ is simply laced, where} \\ S \text{ is a lattice if } \operatorname{rank}(R) > 1, \end{gathered} \tag{1.7}$$

and

$$\begin{gathered} R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \quad \text{if } X \text{ is not simply laced, where} \\ kS + L \subseteq L \subseteq S + L \subseteq S \quad \text{in which} \\ k = 3 \text{ if } X = G_2 \text{ and } k = 2 \text{ otherwise.} \\ \text{Further if } R \text{ has type } G_2 \text{ or } F_4, \text{ then } S \text{ and } L \text{ are lattices.} \\ \text{If } R \text{ has type } B_\ell (l \geq 3), L \text{ is a lattice.} \\ \text{Finally if } R \text{ is of type } C_\ell (l \geq 3), S \text{ is a lattice.} \end{gathered} \tag{1.8}$$

If R has the form (1.7) or (1.8), we write

$$\begin{gathered} R_{sh} := \dot{R}_{sh} + S \quad \text{and} \quad R_{lg} := \dot{R}_{lg} + L \quad \text{and so} \\ R = R^0 \cup R_{sh} \cup R_{lg} \quad \text{and} \quad R^\times = R_{sh} \cup R_{lg}. \end{gathered}$$

We note that the choice of the finite root system $\dot{R}$, defined by (1.2), is not unique. Indeed, it depends on the choice of the preimages of roots in $\bar{\Pi}$ under the map $\bar{}$. The type is however unique. In the next lemma we examine this situation. For a subset S of a vector space $\mathcal{U}$, we denote by $\langle S\rangle$, the subgroup of $\mathcal{U}$ generated by S.

Lemma 1.9 *Let $\dot{R}'$ be another choice of the finite root system appearing in the structure of R. Let*

$$(S+S)\cup(\dot{R}+S)=R=(S'+S')\cup(\dot{R}'+S') \text{ or}$$
$$(S+S)\cup(\dot{R}_{sh}+S)\cup(\dot{R}_{lg}+L)=R=(S'+S')\cup(\dot{R}'_{sh}+S')\cup(\dot{R}'_{lg}+L')$$

be the corresponding expression of R in the form (1.7) or (1.8), according to the type of R and with respect to the finite root systems $\dot{R}$ and $\dot{R}'$, respectively. Then $S'=S+\delta_0$ and $L'=L+\lambda_0$ for some $\delta_0\in S$ and $\lambda_0\in L$.

Proof. First assume that R is not simply laced. So

$(\dot{R}_{sh}+S)=R_{sh}=(\dot{R}'_{sh}+S')$, $(S+S)=R^0=(S'+S')$ and $(\dot{R}_{lg}+L)=R_{lg}=(\dot{R}'_{lg}+L')$.

Therefore $\langle S\rangle=\langle S+S\rangle=\langle S'+S'\rangle=\langle S'\rangle$. Now let $\alpha\in\dot{R}_{sh}\subseteq R_{sh}=(\dot{R}'_{sh}+S')$. Then $\alpha=\alpha'+\delta_0$ for some $\alpha'\in\dot{R}'_{sh}$ and $\delta_0\in S'$. Therefore

$$\alpha-\delta_0+S'=\alpha'+S'\in\dot{R}'_{sh}+S'=R_{sh}=\dot{R}_{sh}+S.$$

Thus $-\delta_0+S'\subseteq S$ or $S'\subseteq S+\delta_0$. On the other hand,

$$\alpha'+\delta_0+S=\alpha+S\in\dot{R}_{sh}+S=R_{sh}=\dot{R}'_{sh}+S'.$$

Thus $\delta_0+S\subseteq S'$ and so $S'=S+\delta_0$. Since $2S'+S'\subseteq S'$ and $\delta_0\in S'$ we get $2\delta_0\in S'$ and so $\delta_0\in S$.

Starting from a root in $\dot{R}_{lg}$ and following a similar pattern as above we get $L'=L+\lambda_0$ for some $\lambda_0\in L$. If R is simply laced, then starting from a root in $\dot{R}\setminus\{0\}$ and a similar argument as above we get $S'=S+\delta_0$ for some $\delta_0\in S$. ☐

Let T be a linear subspace of $\mathcal{V}^0$. By $(\ldots)_T$ we denote the induced bilinear form on the quotient space $\mathcal{V}/T$ defined by $(u+T,v+T)_T:=(u,v)$. Clearly $(\ldots)_T$ is positive semidefinite on $\mathcal{V}/T$. Let

$$R_T:=\{\alpha+T:\alpha\in R\}. \qquad (1.10)$$

Proposition 1.11 *Let R be a reduced EARS and T be a subspace of $\mathcal{V}^0$ so that $\langle R\rangle \cap T$ is a lattice in T. Then R_T is a reduced EARS in $\mathcal{V}/T$ of the same type of R.*

Proof. We must show that R_T satisfies (R1)-(R8). (R1) and (R2) are clear. Since R spans $\mathcal{V}$, $R_T = \{\alpha + T : \alpha \in R\}$ spans $\mathcal{V}/T$, so (R3) holds. For (R4) assume $(\alpha + T) \in R_T$, where $\alpha \in R^\times$ and $2(\alpha + T) \in R_T$. So $2\alpha + T = \alpha_0 + T$ for some $\alpha_0 \in R^\times$. Thus $2\alpha - \alpha_0 \in T \subseteq \mathcal{V}^0$. So $2\alpha + \mathcal{V}^0 = \alpha_0 + \mathcal{V}^0 \in \{\alpha + \mathcal{V}^0 : \alpha \in R\} = \bar{R}$. Since $\bar{R}$ is reduced and $\bar{\alpha} \in \bar{R}$, we get a contradiction. Thus (R4) holds. For (R5) consider the linear projection map $\pi : \mathcal{V} \to \mathcal{V}/T$. Then $R_T = \pi(R)$. By assumption, both $\langle R\rangle$ and $\langle R\rangle \cap T$ are lattices in $\mathcal{V}$ and T, respectively. Therefore, $\mathbf{R} \otimes_{\mathbf{Z}} \langle R\rangle \cong \mathcal{V}$ and $\mathbf{R} \otimes_{\mathbf{Z}} (\langle R\rangle \cap T) \cong T$, where both of these isomorphisms are induced by natural maps. Thus

$$\mathbf{R} \otimes_{\mathbf{Z}} \langle R_T\rangle = \mathbf{R} \otimes_{\mathbf{Z}} \pi(\langle R\rangle) \cong \mathbf{R} \otimes_{\mathbf{Z}} \frac{\langle R\rangle}{\langle R\rangle \cap T} \cong \frac{\mathbf{R} \otimes_{\mathbf{Z}} \langle R\rangle}{\mathbf{R} \otimes_{\mathbf{Z}} (\langle R\rangle \cap T)} \cong \frac{\mathcal{V}}{T}.$$

Therefore $\langle R_T\rangle$ is a lattice in $\mathcal{V}/T$. Hence R_T as a subset of a discrete set is discrete. For $(R6)$, let $\alpha + T, \beta + T \in R_T$ with $\alpha, \beta \in R$ and $(\alpha, \alpha) \neq 0$. Applying (R6) for$R$, we get nonnegative integers u, d such that

$$\{\beta + n\alpha : n \in \mathbf{Z}\} \cap R = \{\beta + n\alpha : -d \leq n \leq u\} \quad \text{with } d - u = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}. \tag{1.12}$$

We are done if we show that

$$\{(\beta + T) + n(\alpha + T) : n \in \mathbf{Z}\} \cap R_T = \{(\beta + T) + n(\alpha + T) : -d \leq n \leq u\}.$$

First note that if $-d \leq m, n \leq u$ and $(\beta + T) + n(\alpha + T) = (\beta + T) + m(\alpha + T)$, then $(n - m)\alpha \in T$. Since α is not isotropic and $T \subseteq \mathcal{V}^0$, we get $n = m$. Thus the set $\{(\beta + T) + n(\alpha + T) : -d \leq n \leq u\}$ has cardinality $d - u + 1 = 1 + 2(\alpha, \beta)/(\alpha, \alpha)$. Now if $-d \leq n \leq u$, then by (1.12), $\beta + n\alpha \in R$, so $(\beta + T) + n(\alpha + T) = (\beta + n\alpha) + T \in R_T$. Therefore

$$\{(\beta + T) + n(\alpha + T) : -d \leq n \leq u\} \subseteq \{(\beta + T) + n(\alpha + T) : n \in \mathbf{Z}\} \cap R_T.$$

So we only need to show that the later set has cardinality $1 + 2(\alpha, \beta)/(\alpha, \alpha)$. For this, let $(\beta + T) + n(\alpha + T) \in R_T$, $\alpha, \beta \in R$, $n \in \mathbf{Z}$. Then we have $\bar{\beta} + n\bar{\alpha} = (\beta + \mathcal{V}^0) + n(\alpha + \mathcal{V}^0) \in R/\mathcal{V}^0 = \bar{R}$. Since $\bar{R}$ is a finite root system and $\bar{\alpha}, \bar{\beta} \in \bar{R}$, the number of such integers n

is equal to $1 + 2(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha}) = 1 + 2(\alpha, \beta)/(\alpha, \alpha)$. Thus (R6) is satisfied. (R7) and (R8) follows easily from the facts that (R7) and (R8) are satisfied for R and that $T \subseteq \mathcal{V}^0$. Since $(\mathcal{V}/T)/(\mathcal{V}^0/T) \cong \mathcal{V}/\mathcal{V}^0$ it follows that $\bar{R} \cong \bar{R_T}$ and so R_T is reduced and has the same type as $\bar{R}$, which is by definition the type of R. □

Before finishing this section, we would like to record some of the facts about semilattices (see Definition I.1.23) which we will need in future sections.

Proposition 1.13 *[AABGP, II.1.4] Suppose that S is a subset of $\mathcal{U}$. If S is a semilattice in $\mathcal{U}$ and $\Lambda = \langle S \rangle$, then Λ is a lattice in $\mathcal{U}$ so that*

$$2\Lambda \subseteq S \subseteq \Lambda, \quad \textit{and} \quad 2\Lambda + S \subseteq S. \tag{1.14}$$

Conversely, if there exists a lattice Λ in $\mathcal{U}$ so that (1.14) holds, then S is a semilattice in $\mathcal{U}$.

It follows from this that (see [AABGP, II.1.7])

$$\text{if } \nu = 1, \text{ then any semilattice in } \mathcal{U} \text{ is a lattice in } \mathcal{U}. \tag{1.15}$$

It also follows that (see [AABGP, II.1.11])

$$\text{the } \mathbb{Z}\text{-span of a semilattice } S \text{ has a } \mathbb{Z}\text{-basis consisting of elements of } S. \tag{1.16}$$

Remark 1.17 *Condition (1.14) can be given a simple interpretation. Namely, given a lattice Λ in $\mathcal{U}$, (1.14) is equivallent to saying that S is the union of a set of cosets of 2Λ in Λ including the trivial coset 2Λ. Therefore there is some nonnegative integer m so that*

$$S = \cup_{i=0}^{m}(\tau_i + 2\Lambda), \quad \textit{where } \tau_i = 0 \textit{ and } \tau_i\textit{'s represent distinct cosets of } 2\Lambda \textit{ in } \Lambda. \tag{1.18}$$

Clearly m is the smallest nonnegative integer so that (1.18) holds. Moreover if $\Lambda = \sum_{i=1}^{\nu} \mathbb{Z}\delta_i$ and $\delta_1, \ldots \delta_\nu \in S$, then $m \geq \nu$ and we can assume that $\tau_1 = \delta_1, \ldots, \tau_\nu = \delta_\nu$.

Definition 1.19 *Let S be a semilattice in $\mathcal{U}$ and $\Lambda = \langle S \rangle$. We define index of S, written $ind(S)$, to be the nonnegative integer m, so that (1.18) holds. Then $dim\mathcal{U} \leq ind(S) \leq 2^{dim\mathcal{U}} - 1$. We also have that any two lattices of the same rank have the same index.*

Lemma 1.20 *Let S be a semilattice and $\delta \in S$. Then $S+\delta$ is also a semilattice and $ind(S+\delta) = ind(S)$.*

Proof. For the first statement see [AABGP, II.1.9]. To see the second statement let $S = \cup_{i=0}^{ind(S)}(\tau_i + 2\langle S\rangle)$ be an expression of S in the form (1.18). Then $S+\delta = \cup_{i=0}^{ind(S)}(\tau_i+\delta+2\langle S\rangle)$ is an expression of $S+\delta$ in the form (1.18). Thus $ind(S+\delta) = ind(S)$. □

Lemma 1.21 *Let S, S_1 and S_2 be semilattices and $S = S_1 \oplus S_2$. Then*

$$ind(S) = ind(S_1)ind(S_2) + ind(S_1) + ind(S_2).$$

Proof. Let

$$S_1 = \cup_{i=0}^{ind(S_1)}(\tau_i + 2\langle S_1\rangle) \quad \text{and} \quad S_2 = \cup_{j=0}^{ind(S_2)}(\sigma_j + 2\langle S_2\rangle)$$

be expressions of S_1 and S_2 in the form (1.18). Then

$$S = S_1 \oplus S_2 = \bigcup_{\substack{0 \le i \le ind(S_1) \\ 0 \le j \le ind(S_2)}} (\tau_i + \sigma_j + 2\langle S\rangle). \tag{1.22}$$

We show that $\tau_i + \sigma_j$, $0 \le i \le ind(S_1)$ and $0 \le j \le ind(S_2)$, represent distinct cosets of $2\langle S\rangle$ in $\langle S\rangle$. If $\tau_i+\sigma_j+2\langle S\rangle = \tau_k+\sigma_\ell+2\langle S\rangle$ for some $0 \le i \le ind(S_1)$ and $0 \le j \le ind(S_2)$, then

$$(\tau_i - \tau_k) + (\sigma_j - \sigma_\ell) \in 2\langle S\rangle = 2\langle S_1\rangle \oplus 2\langle S_2\rangle.$$

On the other hand, $\tau_i - \tau_k \in \langle S_1\rangle$ and $\sigma_j - \sigma_\ell \in \langle S_2\rangle$. Therefore $\tau_i - \tau_k \in 2\langle S_1\rangle$ and $\sigma_j - \sigma_\ell \in 2\langle S_2\rangle$. By the choice of τ_i's and σ_j's, we get $i = k$ and $j = l$. Thus (1.22) gives an expression of S in the form (1.18). So

$$ind(S) = (ind(S_1)+1)(ind(S_2)+1) - 1 = ind(S_1)ind(S_2) + ind(S_1) + ind(S_2). \square$$

Remark 1.23 *If $\mathcal{V}^0$ has dimension 1, then the root systems given in parts (a) and (b) of Construction I.1.24 gives exactly the reduced affine root systems. This is because in this case both S and L are lattices (see (1.15)).*

Definition 1.24 *Suppose (S, L) is a pair of semilattices in $\mathcal{V}^0$ satisfying (1.8). Then we have $kS \subseteq L \subseteq S$. Hence*

$$k\langle S\rangle \subseteq \langle L\rangle \subseteq \langle S\rangle. \tag{1.25}$$

Thus $\langle S\rangle/\langle L\rangle$ *can be identified with a quotient space of the vector space* $\langle S\rangle/k\langle S\rangle$ *over* $\mathbf{F}_k := \mathbb{Z}/k\mathbb{Z}$. *Hence*

$$|\langle S\rangle/\langle L\rangle| = k^t, \quad \text{where } 0 \leq t \leq \nu.$$

The integer t *is called the twist number of pair* (S, L). *If* $R = R(X, S, L)$ *is as in Construction I.1.24(b), we define the twist number of* R *to be the twist number of* (S, L).

Lemma 1.26 *Let* (S, L) *be a pair of semilattices in* $\mathcal{V}^0$ *satisfying (1.8) with twist number* t. *Then there are subspaces* $\mathcal{V}_1^0$ *and* $\mathcal{V}_2^0$ *of* $\mathcal{V}^0$ *and semilattices* S_1 *and* S_2 *of* $\mathcal{V}_1^0$ *and* $\mathcal{V}_2^0$ *respectively, so that*

$$\begin{gathered} \mathcal{V}^0 = \mathcal{V}_1^0 \oplus \mathcal{V}_2^0, \quad dim\mathcal{V}_1^0 = t, \quad dim\mathcal{V}_2^0 = \nu - t, \\ S = S_1 \oplus \langle S_2\rangle \quad \text{and} \quad L = k\langle S_1\rangle \oplus S_2. \end{gathered} \tag{1.27}$$

Proof. For the case $k = 2$, see Proposition 4.17 of [AABGP]. The case $k = 3$ follows easily from Proposition 4.15 of [AABGP]. □

2 Extended Affine Weyl Groups

We recall that R is a reduced EARS. Therefore R has the form (1.7) or has the form (1.8).

Convention 2.1 *The twist number t and the integer k are defined for nonsimply laced cases in Definition 1.24 and (1.8). To be able to have a single argument for both simply laced types and nonsimply laced types we freely talk about semilattices S and L, twist number t and integer k where we assume that*

$$\text{if } X \text{ is simply laced then} \quad \dot{R}_{lg} = \emptyset, \quad t = \nu \quad \text{and} \quad k = 1. \tag{2.2}$$

Also we assume that (1.8) holds only when X is nonsimply laced. Therefore, from now on, no matter what the type, X, of R is, we write

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L). \tag{2.3}$$

Let $(\mathcal{V}^0)^*$ be the real dual space of $\mathcal{V}^0$. We define the bilinear form $(\cdot, \cdot)$ on the $l + 2\nu$ dimensional vector space

$$\tilde{\mathcal{V}} := \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^* \tag{2.4}$$

so that

$$\begin{aligned} &\bullet (\cdot, \cdot) \text{ extends the form } (\cdot, \cdot) \text{ on } \mathcal{V}, \\ &\bullet (\dot{\mathcal{V}}, (\mathcal{V}^0)^*) = \{0\}, \quad ((\mathcal{V}^0)^*, (\mathcal{V}^0)^*) = \{0\}, \\ &\bullet (\mathcal{V}^0, (\mathcal{V}^0)^*) \text{ is the natural pairing.} \end{aligned} \tag{2.5}$$

Clearly the form $(\cdot, \cdot)$ is nondegenerate on $\tilde{\mathcal{V}}$. We normalize the bilinear form so that

$$(\alpha, \alpha) = 2 \text{ if } \alpha \in R_{sh} \quad \text{and} \quad (\alpha, \alpha) = 2k \text{ if } \alpha \in R_{lg}. \tag{2.6}$$

If

$$\check{\alpha} := \frac{2\alpha}{(\alpha, \alpha)} \quad \text{for } \alpha \in \tilde{\mathcal{V}} \text{ with } (\alpha, \alpha) \neq 0,$$

then we have

$$\check{\alpha} = \alpha \text{ if } \alpha \in R_{sh} \quad \text{and} \quad \check{\alpha} = \frac{1}{k}\alpha \text{ if } \alpha \in R_{lg} \tag{2.7}$$

A quick look at the realization of finite root systems (see [H] or [J] for example), gives the following data which we shall use in sequel without any further reference.

Type B_2

$(\alpha,\check\beta)$	$\beta\in R_{sh}$	$\beta\in R_{lg}$
$\alpha\in R_{sh}$	$\{0,\pm2\}$	$\{\pm1\}$
$\alpha\in R_{lg}$	$\{\pm2\}$	$\{0,\pm2\}$

Type $B_\ell(l\geq 3)$

$(\alpha,\check\beta)$	$\beta\in R_{sh}$	$\beta\in R_{lg}$
$\alpha\in R_{sh}$	$\{0,\pm2\}$	$\{0,\pm1\}$
$\alpha\in R_{lg}$	$\{0,\pm2\}$	$\{0,\pm1,\pm2\}$

Type $C_\ell(l\geq 3)$

$(\alpha,\check\beta)$	$\beta\in R_{sh}$	$\beta\in R_{lg}$
$\alpha\in R_{sh}$	$\{0,\pm1,\pm2\}$	$\{0,\pm1\}$
$\alpha\in R_{lg}$	$\{0,\pm2\}$	$\{0,\pm2\}$

Type F_4

$(\alpha,\check\beta)$	$\beta\in R_{sh}$	$\beta\in R_{lg}$
$\alpha\in R_{sh}$	$\{0,\pm1\}$	$\{0,\pm1\}$
$\alpha\in R_{lg}$	$\{0,\pm2\}$	$\{0,\pm1,\pm2\}$

Type G_2

$(\alpha,\check\beta)$	$\beta\in R_{sh}$	$\beta\in R_{lg}$
$\alpha\in R_{sh}$	$\{\pm1,\pm2\}$	$\{0,\pm1\}$
$\alpha\in R_{lg}$	$\{0,\pm3\}$	$\{\pm1,\pm2\}$

Before giving the definition for an extended affine Weyl group we define a notion of duality and follow this with a result.

It is well-known that if $\dot R$ is a finite root system of type X, then $\check{\dot R}=\{\check\alpha:\alpha\in\dot R^\times\}\cup\{0\}$ is a finite root system of type $\check X$, where $\check X$ is the type of the finite root system corresponding to the Dynkin diagram obtained by reversing the arrows of the Dynkin diagram of $\dot R$. $\check{\dot R}$ is called the dual of $\dot R$. One notes that

$$\check{\dot R}\setminus\{0\}=\dot R_{sh}\cup\frac{1}{k}\dot R_{lg}$$

where the integer k depends on the type X of $\dot R$ and is defined as in (1.8) and (2.2). For an extended affine root system R we set

$$\begin{aligned}\check{R}^\times &= \{\check\alpha:\alpha\in R^\times\},\\ \check{R}^0 &= \{\tfrac{2\delta}{(\alpha+\delta,\alpha+\delta)}:\delta\in R^0,\alpha\in R^\times,\alpha+\delta\in R\}\quad\text{and}\\ \check{R} &= \check{R}^\times\cup\check{R}^0\,.\end{aligned}\tag{2.8}$$

Also let

$$\dot{R}_{sh} = \{\dot{\alpha} \mid \alpha \in R_{sh}\} \text{ and } \dot{R}_{lg} = \{\dot{\alpha} \mid \alpha \in R_{lg}\}.$$

Then we have the following result.

Proposition 2.9 *If R has type X, then $\check{R}$ is an EARS of type $\check{X}$. Moreover if X is not simply laced and $R = (S+S) \cup (\dot{R}_{sh}+S) \cup (\dot{R}_{lg}+L)$ is an expression of R in the form (2.3) with nullity ν and twist number t, then $\check{R}$ has nullity ν and twist number $\nu - t$. Furthermore*

$$\check{R} = (\frac{1}{k}L + \frac{1}{k}L) \cup (\frac{1}{k}\dot{R}_{lg} + \frac{1}{k}L) \cup (\dot{R}_{sh} + S) \tag{2.10}$$

Proof. If X is simply laced then from (2.6) we have $\check{R} = R$ and $\check{X} = X$, so we are done. Now suppose X is not simply laced. We first prove that if $R = (S+S) \cup (\dot{R}_{sh}+S) \cup (\dot{R}_{lg}+L)$, then (2.10) holds. Clearly $\check{R_{sh}} = R_{sh}$ and $\check{R_{lg}} = 1/k(R_{lg})$. So it only remains to show that $\check{R^0} = \frac{1}{k}(L+L)$. First we show inclusion "$\subseteq$". Let $\check{\sigma} \in \check{R^0}$. Then $\check{\sigma} = \frac{2\delta}{(\beta+\delta, \beta+\delta)}$ for some $\delta \in R^0$ and $\beta \in R^\times$ with $\beta + \delta \in R$. If $\beta \in R_{sh}$, then $\check{\sigma} = \delta \in R^0 = S + S \subseteq 1/k(L+L)$. If $\beta \in R_{lg}$, then $\beta = \dot{\beta} + \lambda$ for some $\dot{\beta} \in \dot{R}_{lg}$ and $\lambda \in L$. Since $\dot{\beta} + \lambda + \delta \in R$, we have $\lambda + \delta \in L$. So

$$\check{\sigma} = \frac{2\delta}{2k} = \frac{1}{k}\delta \in \frac{1}{k}(L - \lambda) \subseteq \frac{1}{k}(L+L).$$

Thus $\check{R^0} \subseteq \frac{1}{k}(L+L)$. Next, let $\delta \in 1/k(L+L)$. Then $k\delta \in L + L \subseteq S \subseteq R^0$. Therefore there is $\lambda \in L$ so that $k\delta - \lambda \in L$. Let $\dot{\alpha} \in \dot{R}_{lg}$ and $\beta = \dot{\alpha} - \lambda$. Then $\beta \in R_{lg}$ and $\beta + k\delta = \dot{\alpha} + (k\delta - \lambda) \in R_{lg}$. So by definition of $\check{R^0}$ we have $\delta = 2k\delta/(\beta + k\delta, \beta + k\delta) \in \check{R^0}$. Hence (2.10) holds. By Theorem 1.28, $\check{R}$ is an EARS if S and $(1/k)L$ are semilattices in $\mathcal{V}^0$ so that

$$\frac{1}{k}L + S \subseteq \frac{1}{k}L \quad \text{and} \quad L + S \subseteq S. \tag{2.11}$$

But we already know that S and L(and so $(1/k)L$) are semilattices in $\mathcal{V}^0$. Also the inclusions in (2.11) follows immediately from (1.8). To conclude, it only remains to show that $\check{R}$ has type $\check{X}$, nullity ν and twist number $\nu - t$. By Definition I.1.16, $\check{R}$ has nullity ν and by Definition 1.24, $\check{R}$ has twist number t' where $k^{t'} = |\langle(1/k)L\rangle/\langle S\rangle|$. But $t' = \nu - t$ since $k^t = |\langle S\rangle/\langle L\rangle|$. From (2.10) we see that $\check{R}$ has the same type of root system as $\dot{R}_{sh} \cup (1/k)\dot{R}_{lg} \cup \{0\} = \check{\dot{R}}$. Thus it has type $\check{X}$. This completes the proof. □

Remark 2.12 *The notion of duality can be defined similarly for EARS's of type BC. Indeed, if R is an EARS of type $BC_\ell (l \geq 1)$ of nullity ν and twist-triple (t_1, t, t_2) (see Definition 4.34 of [AABGP, II]) and $\check{R}$ is defined by (2.8), then $\check{R}$ is an EARS of type BC_ℓ of nullity ν and twist-triple $(\nu - t_2, \nu + t - t_1 - t_2, \nu - t_1)$. (See [A], Appendix).*

For $\alpha \in \tilde{\mathcal{V}}$ with $(\alpha, \alpha) \neq 0$ we define the reflection $r_\alpha \in GL(\tilde{\mathcal{V}})$ by

$$r_\alpha(\lambda) = \lambda - (\lambda, \check{\alpha})\alpha. \tag{2.13}$$

Lemma 2.14 *(i) The bilinear form $(\cdot,\cdot)$ is invariant under reflections r_α, $\alpha \in \tilde{\mathcal{V}}$, α nonisotropic.*
(ii) $r_\beta r_\alpha r_\beta = r_{r_\beta(\alpha)}$ for any $\alpha, \beta \in \tilde{\mathcal{V}}$, α, β nonisotropic.

Proof. (i) This is easy to see by a straightforward computation. For (ii) note that we have $r_\beta(\check{\alpha}) = (r_\beta(\alpha))\check{}$. Now using part (i) we have, for $\lambda \in \tilde{\mathcal{V}}$,

$$r_\beta r_\alpha r_\beta(\lambda) = r_\beta(r_\beta(\lambda) - (r_\beta(\lambda), \check{\alpha})\alpha) = \lambda - (\lambda, (r_\beta(\alpha))\check{})r_\beta(\alpha) = r_{r_\beta(\alpha)}(\lambda). \qquad \square$$

Definition 2.15 *The extended affine Weyl group $\mathcal{W}_R$ (EAWG) of R is defined to be the subgroup of $GL(\tilde{\mathcal{V}})$ generated by reflections r_α, $\alpha \in R^\times$. When there is no confusion we simply write $\mathcal{W}$ instead of $\mathcal{W}_R$. As we have seen in I.1.53, $\mathcal{W}$ is isomorphic to to the Weyl group of a tame nondegenerate EALA which has R as its root system.*

Note that from (R6) we have

$$\mathcal{W}R \subseteq R. \tag{2.16}$$

Proposition 2.17 *Let R' be another EARS in $\mathcal{V}$. If $R \cong R'$, then $\mathcal{W}_R \cong \mathcal{W}_{R'}$.*

Proof. Since $R \cong R'$, there is $\varphi \in GL(\mathcal{V})$ so that φ preserves the bilinear form $(\cdot,\cdot)$ up to a nonzero scalar and $\varphi(R) = R'$. Since $(v, v) = 0 \Leftrightarrow (\varphi(v), \varphi(v)) = 0$, $v \in \mathcal{V}$, we get $\varphi(\mathcal{V}^0) = \mathcal{V}^0$ and $\varphi(R^\times) = R'^\times$. Thus $\varphi|_{\mathcal{V}^0} \in GL(\mathcal{V}^0)$. Let $\tilde{\mathcal{V}}$ be defined by (2.4). Let $(\cdot,\cdot)$ be the extended bilinear form on $\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}$ as in (2.5). Then one can see that φ can be extended to a map in $GL(\tilde{\mathcal{V}})$, denoted again by φ, so that φ preserves the extended form up to a nonzero scalar. We have $\mathcal{W}_R = \langle r_\alpha : \alpha \in R^\times\rangle$ and $\mathcal{W}_{R'} = \langle r_{\alpha'} : \alpha' \in R'^\times\rangle$. We define $\bar{\varphi} : \mathcal{W}_R \to \mathcal{W}_{R'}$ by

$$\bar{\varphi}(r_{\alpha_1} \cdots r_{\alpha_k}) = r_{\varphi(\alpha_1)} \cdots r_{\varphi(\alpha_k)} \quad \text{for any } \alpha_1, \ldots \alpha_k \in R^\times,\ k \in \mathbb{Z}_{>0}.$$

We prove that $\bar{\varphi}$ defines a group isomorphism. To show that $\bar{\varphi}$ is well-defined and one to one, it is enough to show that

$$r_{\alpha_1}\cdots r_{\alpha_k} = 1 \Leftrightarrow r_{\varphi(\alpha_1)}\cdots r_{\varphi(\alpha_k)} = 1 \quad \text{for any } \alpha_1,\ldots\alpha_k \in R^\times,\ k \in \mathbf{Z}_{>0}.$$

Fix $\alpha_1,\ldots,\alpha_k \in R^\times$. Note that $(\alpha,\check{\beta}) = (\varphi(\alpha),\varphi(\check{\beta}))$ for any $\alpha,\beta \in \tilde{\mathcal{V}}$, β nonisotropic. Using induction on the length of elements of $\mathcal{W}_R$ (an element $w = r_{\alpha_1}\cdots r_{\alpha_n}$, α_i's in $R^\times$ has length n, if n is the smallest positive integer such that w can be written in this form), one can see that, for $\lambda \in \tilde{\mathcal{V}}$,

$$r_{\varphi(\alpha_1)}\cdots r_{\varphi(\alpha_k)}\varphi(\lambda) = \varphi(r_{\alpha_1}\cdots r_{\alpha_k}(\lambda)).$$

Thus $\bar{\varphi}$ is well-defined and one to one. Clearly $\bar{\varphi}$ is a homomorphism. Since $\varphi(R^\times) = R'^\times$, $\bar{\varphi}$ is onto. Thus $\bar{\varphi}$ is an isomorphism. □

The following lemma will be used in the sequel in different places.

Lemma 2.18 *(i) If $X = A_1$, then $\mathcal{W}\dot{R}^\times \subsetneq \dot{R}^\times + 2\langle S\rangle$.*

(ii) If $X = B_\ell (l \geq 2)$, then $\mathcal{W}\dot{R}_{sh} \subseteq \dot{R}_{sh} + \langle L\rangle$.

(iii) If $X = B_2$ or $X = C_\ell (l \geq 3)$, then $\mathcal{W}\dot{R}_{lg} \subseteq \dot{R}_{lg} + 2\langle S\rangle$.

Proof. (i) Since $\mathcal{W}$ is generated by reflections r_α, $\alpha \in R^\times$ and $\mathcal{W}$ fixes pointwise $\mathcal{V}^0$, it is enough to show that $r_\alpha(\dot{R}^\times) \subseteq \dot{R}^\times + 2\langle S\rangle$ for all $\alpha \in R^\times$. So let $\alpha = \dot{\alpha} + \delta \in R^\times$, $\dot{\alpha} \in \dot{R}^\times$, $\delta \in S$. Then

$$r_\alpha(\pm\dot{\alpha}) = \pm(\dot{\alpha} - 2(\dot{\alpha}+\delta)) = \pm(-\dot{\alpha} - 2\delta) \in \dot{R}^\times + 2\langle S\rangle.$$

Since $\dot{R}^\times = \{\dot{\alpha}, -\dot{\alpha}\}$ we are done.

(ii) We only need to show that $r_\alpha(\dot{R}_{sh}) \subseteq \dot{R}_{sh} + \langle L\rangle$ for all $\alpha \in R^\times$. So let $\alpha \in R^\times = R_{sh} \cup R_{lg}$. If $\alpha \in R_{sh}$, then $\alpha = \dot{\alpha} + \delta$, $\dot{\alpha} \in \dot{R}_{sh}$, $\delta \in S$. Then

$$r_\alpha(\dot{R}_{sh}) = r_{\dot{\alpha}+\delta}(\dot{R}_{sh}) \subseteq \dot{R}_{sh} - (\dot{R}_{sh},\check{\dot{\alpha}})\delta \subseteq \dot{R}_{sh} + 2\mathbf{Z}\delta \subseteq \dot{R}_{sh} + 2\langle S\rangle \subseteq \dot{R}_{sh} + \langle L\rangle.$$

If $\alpha \in R_{lg}$, then $\alpha = \dot{\alpha} + \delta$, $\dot{\alpha} \in \dot{R}_{lg}$, $\delta \in L$. So

$$r_\alpha(\dot{R}_{sh}) = r_{\dot{\alpha}+\delta}(\dot{R}_{sh}) \subseteq \dot{R}_{sh} - (\dot{R}_{sh},\check{\dot{\alpha}})\delta \subseteq \dot{R}_{sh} + \mathbf{Z}\delta \subseteq \dot{R}_{sh} + \langle L\rangle.$$

(iii) Again we only need to show that $r_\alpha(\dot{R}_{lg}) \subseteq \dot{R}_{lg} + 2\langle S\rangle$ for all $\alpha \in R^\times = R_{sh} \cup R_{lg}$. If $\alpha \in R_{sh}$, then $\alpha = \dot{\alpha} + \delta$, $\dot{\alpha} \in \dot{R}_{sh}$, $\delta \in S$. Then

$$r_\alpha(\dot{R}_{lg}) = r_{\dot{\alpha}+\delta}(\dot{R}_{lg}) \subseteq \dot{R}_{lg} - (\dot{R}_{lg}, \check{\dot{\alpha}})\delta \subseteq \dot{R}_{lg} + 2\mathbb{Z}\delta \subseteq \dot{R}_{lg} + 2\langle S\rangle.$$

If $\alpha \in R_{lg}$, then $\alpha = \dot{\alpha} + \delta$, $\dot{\alpha} \in \dot{R}_{lg}$, $\delta \in L$. So

$$r_\alpha(\dot{R}_{lg}) = r_{\dot{\alpha}+\delta}(\dot{R}_{lg}) \subseteq \dot{R}_{lg} - (\dot{R}_{lg}, \check{\dot{\alpha}})\delta \subseteq \dot{R}_{lg} + 2\mathbb{Z}\delta \subseteq \dot{R}_{lg} + 2\langle L\rangle \subseteq \dot{R}_{lg} + 2\langle S\rangle. \square$$

Now let R' be another EARS in $\mathcal{V}$ of the form

$$R' = (S' + S') \cup (\dot{R} + S') \quad \text{with} \quad \langle S'\rangle = \langle S\rangle \tag{2.19}$$

if X is simply laced and

$$\begin{gathered} R' = (S' + S') \cup (\dot{R}_{sh} + S') \cup (\dot{R}_{lg} + L'). \\ S' + L' \subseteq S' \text{ and } kS' + L' \subseteq L' \text{ with} \\ \langle S'\rangle = \langle S\rangle \quad \text{and} \quad \langle L'\rangle = \langle L\rangle. \end{gathered} \tag{2.20}$$

if X is nonsimply laced and reduced type. Using Convention 2.1 we can always write R' in the form (2.20). Then we have

Lemma 2.21 *If $R \subseteq R'$ then $\mathcal{W}_{R'}R^\times \subseteq R^\times$.*

Proof. We consider the following cases:

(1) $X = A_1$. By Lemma 2.18(replacing R with R' and $\mathcal{W}_R$ with $\mathcal{W}_{R'}$), we have

$$\mathcal{W}_{R'}R^\times = \mathcal{W}_{R'}(\dot{R}^\times + S) \subseteq \dot{R}^\times + 2\langle S'\rangle + S = \dot{R}^\times + 2\langle S\rangle + S \subseteq \dot{R}^\times + S = R^\times.$$

(2) X is simply laced with rank > 1. By Theorem I.1.28, in this case S is a lattice, so $S = \langle S\rangle = \langle S'\rangle = S'$. Thus $R = R'$ and so

$$\mathcal{W}_{R'}R^\times = \mathcal{W}_R R^\times \subseteq R^\times.$$

(3) $X = B_\ell (l \geq 2)$. We need to show $\mathcal{W}_{R'}R_{sh} \subseteq R_{sh}$ and $\mathcal{W}_{R'}R_{lg} \subseteq R_{lg}$. By Lemma 2.18 we have

$$\mathcal{W}_{R'}R_{sh} = \mathcal{W}_{R'}(\dot{R}_{sh} + S) \subseteq \dot{R}_{sh} + \langle L'\rangle + S = \dot{R}_{sh} + \langle L\rangle + S \subseteq \dot{R}_{sh} + S = R_{sh}.$$

To show $\mathcal{W}_{R'}R_{lg} \subseteq R_{lg}$ we note that if $l \geq 3$, then by Theorem I.1.28, $\langle L\rangle = \langle L'\rangle$ and so $R_{lg} = R'_{lg}$. Then

$$\mathcal{W}_{R'}R_{lg} = \mathcal{W}_{R'}R'_{lg} \subseteq R'_{lg} = R_{lg}.$$

So it only remains to consider the case $l = 2$. If $l = 2$, then Lemma 2.18 gives

$$\mathcal{W}_{R'}R_{lg} = \mathcal{W}_{R'}(\dot{R}_{lg} + L) \subseteq \dot{R}_{lg} + 2\langle S'\rangle + L = \dot{R}_{lg} + 2\langle S\rangle + L \subseteq \dot{R}_{lg} + L = R_{lg}.$$

(4) $X = C_\ell(l \geq 3)$. By Theorem I.1.28, $\langle S\rangle = \langle S'\rangle$ and so $R_{sh} = R'_{sh}$. Thus

$$\mathcal{W}_{R'}R_{sh} \subseteq \mathcal{W}_{R'}R'_{sh} \subseteq R'_{sh} = R_{sh}.$$

Also by Lemma 2.18, we have

$$\mathcal{W}_{R'}R_{lg} = \mathcal{W}_{R'}(\dot{R}_{lg} + L) \subseteq \dot{R}_{lg} + 2\langle S'\rangle + L = \dot{R}_{lg} + 2\langle S\rangle + L \subseteq \dot{R}_{lg} + L = R_{lg}.$$

(5) $X = F_4$ or G_2. By Theorem I.1.28, S, S', L and L' are lattices. Therefore $R = R'$ and so $\mathcal{W}_{R'}R^\times = \mathcal{W}_R R^\times \subseteq R^\times$. □

We will use the symbole "$\lhd$" for the notion of "normal subgroup".

Proposition 2.22 *Let R' be an EARS in $\mathcal{V}$ as in (2.20). If $R \subseteq R'$ then $\mathcal{W}_R \lhd \mathcal{W}_{R'}$.*

Proof. Since $R \subseteq R'$ we have $S \subseteq S'$, $L \subseteq L'$ and $\mathcal{W}_R \subseteq \mathcal{W}_{R'}$. By Lemma 2.14 the normality of $\mathcal{W}_R$ in $\mathcal{W}_{R'}$ is clear if we show that $\mathcal{W}_{R'}R^\times \subseteq R^\times$. But we have seen this in Lemma 2.21. □

Starting from EARS R we now introduce a new EARS $\tilde{R}$ as follows. If X is simply laced we set

$$\tilde{R} = \langle S\rangle \cup (\dot{R} + \langle S\rangle)$$

and if X is nonsimply laced and reduced we set

$$\tilde{R} = \langle S\rangle \cup (\dot{R}_{sh} + \langle S\rangle) \cup (\dot{R}_{lg} + \langle L\rangle).$$

By Theorem I.1.28, $\tilde{R}$ is an EARS of type X. It is clear that $\tilde{R}$ has the same twist number of R. We denote by $\tilde{\mathcal{W}}$ the EAWG of $\tilde{R}$.

Lemma 2.23 *(i) The bilinear form $(\cdot,\cdot)$ is $\tilde{\mathcal{W}}$-invariant.*

(ii) $wr_\alpha w^{-1} = r_{w\alpha}$ for $\alpha \in \tilde{R}^\times$, $w \in \tilde{\mathcal{W}}$.

(iii) $\mathcal{W} \lhd \tilde{\mathcal{W}}$

Proof. (i) and (ii) follows immediately from Lemma 2.14.

(iii) Since the EARS $\tilde{R}$ is of the form (2.20) and $S \subseteq \langle S \rangle$ and $L \subseteq \langle L \rangle$ therefore by Proposition 2.22, we have $\mathcal{W} \lhd \tilde{\mathcal{W}}$. $\square$

3 Structure of Extended Affine Weyl Groups

Let R be the EARS as in (2.3) and (S, L) be the corresponding pair of semilattices. By Lemma 1.26 we can write $\mathcal{V}^0 = \mathcal{V}_1^0 \oplus \mathcal{V}_2^0$, $S = S_1 \oplus \langle S_2 \rangle$ and $L = k\langle S_1 \rangle \oplus S_2$, where S_1 and S_2 are semilattices in $\mathcal{V}_1^0$ and $\mathcal{V}_2^0$, respectively. By (1.16), $\mathcal{V}_1^0$ has a basis $\delta_1, \ldots \delta_t$ and $\mathcal{V}_2^0$ has a basis $\delta_{t+1}, \ldots, \delta_\nu$ so that

$$\begin{gathered} \delta_1, \ldots, \delta_t \in S_1, \quad \delta_{t+1}, \ldots, \delta_\nu \in S_2, \\ \langle S_1 \rangle = \mathbf{Z}\delta_1 \oplus \cdots \oplus \mathbf{Z}\delta_t \quad \text{and} \quad \langle S_2 \rangle = \mathbf{Z}\delta_{t+1} \oplus \cdots \oplus \mathbf{Z}\delta_\nu. \end{gathered} \tag{3.1}$$

Therefore we can write

$$\langle L \rangle = \sum_{i=1}^{\nu} k_i \mathbf{Z}\delta_i \text{ with } k_i = k \text{ for } 1 \le i \le t \quad \text{and} \quad k_i = 1 \text{ for } t+1 \le i \le \nu. \tag{3.2}$$

Let $\dot{A}$ be the Cartan matrix corresponding to $\dot{R}$ with respect to the fundamental system $\dot{\Pi} := \{\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell\}$. Then

$$\tilde{\mathcal{V}} = \sum_{i=1}^{l} \mathbf{R}\dot{\alpha}_i \oplus \sum_{i=1}^{\nu} \mathbf{R}\delta_i \oplus \sum_{i=1}^{\nu} \mathbf{R}\Lambda_i, \tag{3.3}$$

where $\Lambda_1, \ldots \Lambda_\nu$ is the basis of $(\mathcal{V}^0)^*$ dual to $\delta_1, \ldots, \delta_\nu$.

Here we need to introduce some notation. Let $\mathbf{Z}T$ denote the $\mathbf{Z}$-span of a subset T in $\mathcal{V}$. We set

$$\begin{aligned}
&J = \{1, 2, \ldots, \nu\} \\
&J_s = \{1, \ldots, t\} = \{i \in J : k_i \neq 1\}, \quad J_\ell = J \setminus J_s, \\
&\dot{Q} = \mathbf{Z}\dot{R}, \quad \dot{Q}_s = \mathbf{Z}\dot{R}_{sh}, \quad \dot{Q}_\ell = \mathbf{Z}\dot{R}_{lg}, \\
&Q = \mathbf{Z}\tilde{R} = \mathbf{Z}R = \dot{Q} + \textstyle\sum_{i=1}^{\nu} \mathbf{Z}\delta_i \\
&Q_s = \mathbf{Z}\tilde{R}_{sh} = \mathbf{Z}R_{sh} = \dot{Q}_s + \textstyle\sum_{i=1}^{\nu} \mathbf{Z}\delta_i = \dot{Q}_s + \langle S \rangle, \\
&Q_\ell = \mathbf{Z}\tilde{R}_{lg} = \mathbf{Z}R_{lg} = \dot{Q}_\ell + \textstyle\sum_{i=1}^{\nu} k_i \mathbf{Z}\delta_i = \dot{Q}_\ell + \langle L \rangle \quad \text{and} \\
&\dot{\mathcal{W}} = \langle r_{\dot{\alpha}_i} : 1 \le i \le l \rangle \le GL(\tilde{\mathcal{V}}).
\end{aligned} \tag{3.4}$$

Note that $\dot{\mathcal{W}}$ can be identified as a subgroup of $GL(\dot{\mathcal{V}})$ and so is isomorphic to the finite Weyl group of $\dot{R}$. Also note that

$$|J_s| = t \quad \text{and} \quad |J_\ell| = \nu - t.$$

Lemma 3.5 *If R is nonsimply laced and reduced, then we have*

$$k\dot{Q}_s \subseteq \dot{Q}_\ell \subseteq \dot{Q}_s \quad \textit{and} \quad kQ_s \subseteq Q_\ell \subseteq Q_s.$$

Proof. Checking the Dynkin diagram of $\dot{R}$, one can see that there are simple roots $\alpha \in \dot{R}_{sh}$, $\beta \in \dot{R}_{lg}$ such that $(\alpha, \check{\beta}) = -1$ and $(\beta, \check{\alpha}) = -k$. Thus $\alpha + \beta = r_\beta(\alpha) \in \dot{R}_{sh}$ and $\beta + k\alpha = r_\alpha(\beta) \in \dot{R}_{lg}$. Therefore $\beta \in \dot{R}_{sh} - \alpha \subseteq \dot{Q}_s$ and $k\alpha \in \dot{R}_{lg} - \beta \subseteq \dot{Q}_\ell$. Then we have

$$\dot{R}_{lg} = \dot{\mathcal{W}}\beta \subseteq \dot{\mathcal{W}}\dot{Q}_s = \mathbb{Z}(\dot{\mathcal{W}}\dot{R}_{sh}) \subseteq \mathbb{Z}\dot{R}_{sh} = \dot{Q}_s.$$

This gives $\dot{Q}_\ell \subseteq \dot{Q}_s$. Also

$$k\dot{R}_{sh} = k\dot{\mathcal{W}}\alpha = \dot{\mathcal{W}}(k\alpha) \subseteq \dot{\mathcal{W}}\dot{Q}_\ell \subseteq \mathbb{Z}(\dot{\mathcal{W}}\dot{R}_{lg}) \subseteq \mathbb{Z}\dot{R}_{lg} = \dot{Q}_\ell.$$

This gives $k\dot{Q}_s \subseteq \dot{Q}_\ell$. The second part of the statement now follows from this and the fact that $kS \subseteq L \subseteq S$. □

Let $\alpha \in \mathcal{V}$ be nonisotropic and $\delta \in \mathcal{V}^0$. Then, for $\lambda \in \tilde{\mathcal{V}}$, we have

$$r_{\alpha+\delta}r_\alpha(\lambda) = \lambda - (\lambda, \delta)\check{\alpha} + [(\lambda, \check{\alpha}) - \frac{1}{2}(\check{\alpha}, \check{\alpha})(\lambda, \delta)]\delta. \tag{3.6}$$

In fact

$$\begin{aligned} r_{\alpha+\delta}r_\alpha(\lambda) &= r_{\alpha+\delta}(\lambda - (\lambda, \check{\alpha})\alpha) \\ &= \lambda - (\lambda, \alpha \check{+} \delta)(\alpha + \delta) - (\lambda, \check{\alpha})\alpha + (\lambda, \check{\alpha})(\alpha, \alpha \check{+} \delta)(\alpha + \delta) \\ &= \lambda - (\lambda, \delta)\check{\alpha} + [-(\lambda, \check{\alpha}) - (\lambda, \check{\alpha}) + (\lambda, \check{\alpha})(\alpha, \check{\alpha})]\alpha \\ &\quad +[-(\lambda, \frac{2\delta}{(\alpha, \alpha)}) - (\lambda, \check{\alpha}) + 2(\lambda, \check{\alpha})]\delta \\ &= \lambda - (\lambda, \delta)\check{\alpha} + [(\lambda, \check{\alpha}) - \frac{1}{2}(\check{\alpha}, \check{\alpha})(\lambda, \delta)]\delta. \end{aligned}$$

where the last equality follows from the fact that $(\alpha, \check{\alpha}) = 2$. Therefore if $\check{\alpha} = \frac{1}{k}\alpha$ (in particular if $\alpha \in \tilde{R}_{lg}$) we have

$$r_{\alpha+\delta}r_\alpha(\lambda) = \lambda - (\lambda, \delta)\frac{1}{k}\alpha + [(\lambda, \frac{1}{k}\alpha) - \frac{1}{2}(\frac{1}{k}\alpha, \frac{1}{k}\alpha)(\lambda, \delta)]\delta, \tag{3.7}$$

and if $\check{\alpha} = \alpha$ (in particular if $\alpha \in \tilde{R}_{sh}$) we have

$$r_{\alpha+\delta}r_\alpha(\lambda) = \lambda - (\lambda, \delta)\alpha + [(\lambda, \alpha) - \frac{1}{2}(\alpha, \alpha)(\lambda, \delta)]\delta. \tag{3.8}$$

This leads us to define, for $i \in J$, and $\alpha \in Q$, the linear map $t_\alpha^{(i)} : \tilde{\mathcal{V}} \longrightarrow \tilde{\mathcal{V}}$ given by

$$t_\alpha^{(i)}(\lambda) := \lambda - (\lambda, \frac{1}{k}\delta_i)\alpha + [(\lambda,\alpha) - \frac{1}{2}(\alpha,\alpha)(\lambda, \frac{1}{k}\delta_i)]\frac{1}{k}\delta_i. \tag{3.9}$$

(One may call this an Eichler-Siegel transformation (see [Ei] or [Eb]).) It is easy to see that for all $n \in \mathbf{Z}$, $i \in J$, and $\beta \in Q$,

$$t_{n\delta_i}^{(i)} = 1 \quad \text{and} \quad t_\alpha^{(i)}(\beta) = \beta + (\beta, \frac{1}{k}\alpha)\delta_i. \tag{3.10}$$

We say $(\cdot,\cdot)$ is invariant under a map $T \in GL(\tilde{\mathcal{V}})$ if $(T(\alpha), T(\beta)) = (\alpha,\beta)$ for all $\alpha, \beta \in \tilde{\mathcal{V}}$.

Lemma 3.11 *For $i \in J$, we have*

(i) $t_\alpha^{(i)} t_\beta^{(i)} = t_{\alpha+\beta}^{(i)}$, for all $\alpha, \beta \in Q$. Hence $t_\alpha^{(i)} \in Aut(\tilde{\mathcal{V}})$.

(ii) For any $T \in GL(\tilde{\mathcal{V}})$ satisfying T fixes pointwise $\mathcal{V}^0$ and the bilinear form $(\cdot,\cdot)$ is invariant under T we have $Tt_\alpha^{(i)}T^{-1} = t_{T\alpha}^{(i)}$,

(iii) $wt_\alpha^{(i)}w^{-1} = t_{w\alpha}^{(i)}$, for $\alpha \in Q$, $w \in \tilde{\mathcal{W}}$.

(iv) $t_\alpha^{(j)} t_\beta^{(i)} t_{-\alpha}^{j} = t_{t_\alpha^{(j)}(\beta)}^{(i)}$ for $i, j \in J$, $\alpha, \beta \in Q$.

Proof. We have

$$\begin{aligned}
t_\alpha^{(i)} t_\beta^{(i)}(\lambda) &= t_\alpha^{(i)}(\lambda - (\lambda, \frac{1}{k}\delta_i)\beta + [(\lambda,\beta) - \frac{1}{2}(\beta,\beta)(\lambda,\frac{1}{k}\delta_i)]\frac{1}{k}\delta_i) \\
&= \lambda - (\lambda, \frac{1}{k}\delta_i)\alpha + [(\lambda,\alpha) - \frac{1}{2}(\alpha,\alpha)(\lambda,\frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&\quad -(\lambda,\frac{1}{k}\delta_i)(\beta + \frac{1}{k}(\beta,\alpha)\delta_i) \\
&\quad +[(\lambda,\beta) - \frac{1}{2}(\beta,\beta)(\lambda,\frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&= \lambda - (\lambda,\frac{1}{k}\delta_i)(\alpha+\beta) + (\lambda,\alpha+\beta)\frac{1}{k}\delta_i \\
&\quad -\frac{1}{2}[(\alpha,\alpha) + (\beta,\beta) + 2(\beta,\alpha)](\lambda,\frac{1}{k}\delta_i)\frac{1}{k}\delta_i \\
&= \lambda - (\lambda,\frac{1}{k}\delta_i)(\alpha+\beta) \\
&\quad +[(\lambda,\alpha+\beta) - \frac{1}{2}(\alpha+\beta,\alpha+\beta)(\lambda,\frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&= t_{\alpha+\beta}^{(i)}(\lambda).
\end{aligned}$$

(ii)

$$
\begin{aligned}
&(Tt_\alpha^{(i)}T^{-1})(\lambda)\\
&= T(T^{-1}\lambda - (T^{-1}\lambda, \tfrac{1}{k}\delta_i)\alpha + [(T^{-1}\lambda, \alpha) - \tfrac{1}{2}(\alpha,\alpha)(T^{-1}\lambda, \tfrac{1}{k}\delta_i)]\tfrac{1}{k}\delta_i)\\
&= \lambda - (\lambda, \tfrac{1}{k}\delta_i)T\alpha + [(\lambda, T\alpha) - \tfrac{1}{2}(T\alpha, T\alpha)(\lambda, \tfrac{1}{k}\delta_i)]\tfrac{1}{k}\delta_i\\
&= t_{T\alpha}^{(i)}(\lambda).
\end{aligned}
$$

(iii)-(iv) By Lemma 2.23, the bilinear form $(\cdot,\cdot)$ is $\tilde{\mathcal{W}}$-invariant. It is also straightforward to see that the bilinear form $(\cdot,\cdot)$ is invariant under the linear maps $t_\alpha^{(i)}$, $i \in J$, $\alpha \in Q$. Now both (iii) and (iv) follow immediately from (ii). □

Lemma 3.12 *Let $\alpha \in \tilde{R}^\times$ and $\delta = \sum_{i=1}^{\nu} m_i\delta_i$, $m_i \in \mathbf{Z}$, then*

(i) If $\alpha \in \tilde{R}_{lg}$, we have

$$r_{\alpha+\delta_i}r_\alpha = t_\alpha^{(i)}, \quad \text{and} \quad r_{\alpha+\delta}r_\alpha = (t_{\alpha+\beta_\nu}^{(\nu)})^{m_\nu}(t_{\alpha+\beta_{\nu-1}}^{(\nu-1)})^{m_{\nu-1}}\cdots(t_{\alpha+\beta_1}^{(1)})^{m_1}.$$

where $\beta_1 = 0$ and $\beta_j := \sum_{i=1}^{j-1} m_i\delta_i$, $2 \le j \le \nu$.

(ii) If $\alpha \in \tilde{R}_{sh}$, we have

$$r_{\alpha+\delta_i}r_\alpha = t_{k\alpha}^{(i)} \quad \text{and} \quad r_{\alpha+\delta}r_\alpha = (t_{k(\alpha+\beta_\nu)}^{(\nu)})^{m_\nu}(t_{k(\alpha+\beta_{\nu-1})}^{(\nu-1)})^{m_{\nu-1}}\cdots(t_{k(\alpha+\beta_1)}^{(1)})^{m_1}.$$

where $\beta_1 = 0$ and $\beta_j := \sum_{i=1}^{j-1} m_i\delta_i$, $2 \le j \le \nu$.

Proof. (i) From (3.7), we get $r_{\alpha+\delta_i}r_\alpha = t_\alpha^{(i)}$. Now let $\beta_j := \sum_{i=1}^{j-1} m_i\delta_i$. If $m_i > 0$, then we have

$$
\begin{aligned}
r_{\alpha+m_i\delta_i} &= r_{\alpha+(m_i-1)\delta_i+\delta_i}\\
&= t_{\alpha+(m_i-1)\delta_i}^{(i)} r_{\alpha+(m_i-1)\delta_i}\\
&= t_{\alpha+(m_i-1)\delta_i}^{(i)} t_{\alpha+(m_i-2)\delta_i}^{(i)} r_{\alpha+(m_i-2)\delta_i}\\
&\;\vdots\\
&= t_{\alpha+(m_i-1)\delta_i}^{(i)} t_{\alpha+(m_i-2)\delta_i}^{(i)} \dots t_\alpha^{(i)} r_\alpha.
\end{aligned}
$$

Since $t_{\alpha+n\delta_i}^{(i)} = t_\alpha^{(i)}t_{n\delta_i}^{(i)}$ and $t_{n\delta_i}^{(i)} = 1$ for all $n \in \mathbf{Z}$, then $r_{\alpha+m_i\delta_i}r_\alpha = (t_\alpha^{(i)})^{m_i} = t_{m_i\alpha}^{(i)}$. If $m_i < 0$, then $-m_i > 0$ and $r_{\alpha+m_i\delta_i}r_\alpha = r_{-\alpha-m_i\delta_i}r_{-\alpha} = (t_{-\alpha}^{(i)})^{-m_i} = t_{m_i\alpha}^{(i)}$. Then

$$r_{\alpha+\delta} = r_{\alpha+\beta_\nu+m_\nu\delta_\nu} = (t_{m_\nu(\alpha+\beta_\nu)}^{(\nu)})r_{\alpha+\beta_\nu}$$

$$
\begin{aligned}
&= (t^{(\nu)}_{m_\nu(\alpha+\beta_\nu)})^{m_\nu} r_{\alpha+\beta_{\nu-1}+m_{\nu-1}\delta_{\nu-1}} \\
&= (t^{(\nu)}_{m_\nu(\alpha+\beta_\nu)})^{m_\nu} (t^{(\nu-1)}_{m_{\nu-1}(\alpha+\beta_{\nu-1})})^{m_{\nu-1}} r_{\alpha+\beta_{\nu-1}} \\
&\vdots \\
&= (t^{(\nu)}_{\alpha+\beta_\nu})^{m_\nu} \cdots (t^{(1)}_\alpha)^{m_1} r_\alpha.
\end{aligned}
$$

This finishes the proof of (i).

(ii) Using (3.8) with an analogous argument as in part (i) we get (ii). □

To study the structure of the EAWG $\mathcal{W}$ and $\tilde{\mathcal{W}}$ we need to define some new terms. Let

$$
\begin{aligned}
\tilde{H} &:= \langle r_{\alpha+\delta} r_\alpha : \alpha \in \tilde{R}^\times, \delta \in \mathcal{V}^0, \alpha+\delta \in \tilde{R}^\times \rangle \leq \tilde{\mathcal{W}} \quad \text{and} \\
H &:= \langle r_{\alpha+\delta} r_\alpha : \alpha \in R^\times, \delta \in \mathcal{V}^0, \alpha+\delta \in R^\times \rangle \leq \mathcal{W}.
\end{aligned}
\tag{3.13}
$$

Lemma 3.14 *$\tilde{H} \lhd \tilde{\mathcal{W}}, H \lhd \tilde{\mathcal{W}}$ and $H \lhd \mathcal{W}$.*

Proof. Since $H \subseteq \tilde{H} \subseteq \tilde{\mathcal{W}}$ and $\mathcal{W} \subseteq \tilde{\mathcal{W}}$, we just need to show that H and $\tilde{H}$ are normal in $\tilde{\mathcal{W}}$. From Lemma 2.23(ii) we have

$$
w r_{\alpha+\delta} r_\alpha w^{-1} = w r_{\alpha+\delta} w^{-1} w r_\alpha w^{-1} = r_{w\alpha+\delta} r_{w\alpha},
$$

for any $w \in \tilde{\mathcal{W}}$ and $\alpha \in \tilde{R}^\times$. We have $\tilde{\mathcal{W}}\tilde{R} \subseteq \tilde{R}$, so $w\alpha \in \tilde{R}^\times$ and $w\alpha + \delta = w(\alpha+\delta) \in \tilde{R}^\times$. Thus from the definition of $\tilde{H}$ we have $r_{w\alpha+\delta} r_{w\alpha} \in \tilde{H}$. If $\alpha \in R^\times$, then from Lemma 2.21 we have $w\alpha \in R^\times$ and $w\alpha + \delta = w(\alpha+\delta) \in R^\times$. Thus by definition of H, we have $r_{w\alpha+\delta} r_{w\alpha} \in H$. Hence $\tilde{H} \lhd \tilde{\mathcal{W}}$ and $H \lhd \tilde{\mathcal{W}}$. □

Lemma 3.15 $\tilde{H} = \langle t^{(i)}_\alpha : (i,\alpha) \in (J_\ell \times Q_\ell) \cup (J_s \times kQ_s) \rangle$

Proof. Let us denote by T the right hand side of the statement. First we show that $T \subseteq \tilde{H}$. We do this in the following two steps:

(1) Let $i \in J_\ell$ and $\alpha \in \tilde{R}_{lg} = \dot{R}_{lg} + \sum_{j=1}^{\nu} k_j \mathbf{Z}\delta_j$. Since $k_i = 1$, we have $\alpha + \delta_i \in \tilde{R}_{lg}$. By Lemma 3.12(i), we have $t^{(i)}_\alpha = r_{\alpha+\delta_i} r_\alpha \in \tilde{H}$. Now Lemma 3.11(i) implies that $t^{(i)}_\alpha \in \tilde{H}$, for all $\alpha \in Q_\ell$.

(2) Let $i \in J_s$ and $\alpha \in \tilde{R}_{sh} = \dot{R}_{sh} + \sum_{j=1}^{\nu} \mathbf{Z}\delta_j$. Then $\alpha + \delta_i \in \tilde{R}_{sh}$. By Lemma 3.12(ii), we have $t^{(i)}_{k\alpha} = r_{\alpha+\delta_i} r_\alpha \in \tilde{H}$. Again Lemma 3.11(i) implies that $t^{(i)}_{k\alpha} \in \tilde{H}$ for all $\alpha \in Q_s$. Thus $T \subseteq \tilde{H}$.

We now show that $\tilde{H} \subseteq T$. Let $\alpha \in \tilde{R}^\times$, $\delta = \sum_{i=1}^{\nu} m_i\delta_i$ and $\alpha + \delta \in \tilde{R}^\times$. We must show that $r_{\alpha+\delta}r_\alpha \in T$. First let $\alpha \in \tilde{R}_{lg}$. By Lemma 3.12(i), we have

$$r_{\alpha+\delta}r_\alpha = (t^{(\nu)}_{\alpha+\beta_\nu})^{m_\nu}(t^{(\nu-1)}_{\alpha+\beta_{\nu-1}})^{m_{\nu-1}}\cdots(t^{(1)}_{\alpha+\beta_1})^{m_1}. \tag{3.16}$$

where $\beta_1 = 0$ and $\beta_j = \sum_{i=1}^{j-1} m_i\delta_i$, for $2 \le j \le \nu$. We notice here that since $\tilde{R}_{lg} = \dot{R}_{lg} + \langle L\rangle$ and $\langle L\rangle$ is a lattice we have $\alpha + \beta_j \in \tilde{R}_{lg}$ for all j, $1 \le j \le \nu$. By (3.16), it is enough to show that $(t^{(j)}_{\alpha+\beta_j})^{m_j} \in T$ for $1 \le j \le \nu$. If $j \in J_\ell$, then $(j, \alpha + \beta_j) \in J_\ell \times Q_\ell$ and so $t^{(j)}_{\alpha+\beta_j} \in T$. Thus $(t^{(j)}_{\alpha+\beta_j})^{m_j} \in T$. Now let $j \in J_s$. Then $k_j = k$. Since $\delta \in \langle L\rangle = \sum_{j=1}^{\nu} k_j\mathbf{Z}\delta_j$ we have $m_j \in k\mathbf{Z}$. Let $m_j = km'_j$ for some $m'_j \in \mathbf{Z}$. We have $\alpha + \beta_j \in Q_\ell \subseteq Q_s$, so $(j, k(\alpha+\beta_j)) \in J_s \times kQ_s$. Thus $t^{(j)}_{k(\alpha+\beta_j)} \in T$. This gives

$$(t^{(j)}_{\alpha+\beta_j})^{m_j} = (t^{(j)}_{\alpha+\beta_j})^{km'_j} = (t^{(j)}_{k(\alpha+\beta_j)})^{m'_j} \in T.$$

This takes care of the case in which $\alpha \in \tilde{R}_{lg}$. Now let $\alpha \in \tilde{R}_{sh}$. By Lemma 3.12(ii), we have

$$r_{\alpha+\delta}r_\alpha = (t^{(\nu)}_{k(\alpha+\beta_\nu)})^{m_\nu}(t^{(\nu-1)}_{k(\alpha+\beta_{\nu-1})})^{m_{\nu-1}}\cdots(t^{(1)}_{k(\alpha+\beta_1)})^{m_1}.$$

where $\beta_1 = 0$ and $\beta_j = \sum_{i=1}^{j-1} m_i\delta_i$, for j, $1 \le j \le \nu$. Since $\tilde{R}_{sh} = \dot{R}_{sh} + \langle S\rangle$ and $\langle S\rangle$ is a lattice, $\alpha + \beta_j \in \tilde{R}_{sh}$ for $2 \le j \le \nu$. We are done if we show that $(t^{(j)}_{k(\alpha+\beta_j)})^{m_j} \in T$ for $2 \le j \le \nu$. Now if $j \in J_s$, then $(j, k(\alpha+\beta_j)) \in J_s \times Q_s$ and so $t^{(j)}_{k(\alpha+\beta_j)} \in T$. Then $(t^{(j)}_{k(\alpha+\beta_j)})^{m_j} \in T$. If $j \in J_\ell$, we have $k(\alpha+\beta_j) \in kQ_s \subseteq Q_\ell$ (see Lemma 3.5) and so $(j, k(\alpha+\beta_j)) \in J_\ell \times Q_\ell$. Thus $(t^{j}_{k(\alpha+\beta_j)})^{m_j} \in T$. This completes the proof of Lemma. $\square$

We set

$$\begin{aligned} c_{ij} &= t^{(i)}_{-\delta_j} \quad (i, j \in J), \\ H^{(i)}_\ell &= \langle t^{(i)}_\alpha : \alpha \in \dot{Q}_\ell\rangle \quad (i \in J_\ell), \\ H^{(i)}_s &= \langle t^{(i)}_\alpha : \alpha \in k\dot{Q}_s\rangle \quad (i \in J_s). \end{aligned} \tag{3.17}$$

Then for $i, j \in J$ and $\lambda \in \tilde{\mathcal{V}}$ we have

$$c_{ij}(\lambda) = \lambda - \frac{1}{k}(\lambda, \delta_j)\delta_i + \frac{1}{k}(\lambda, \delta_i)\delta_j, \quad c_{ii} = 1 \quad \text{and} \quad c_{ij}c_{ji} = 1. \tag{3.18}$$

Lemma 3.19 *$H^{(i)}_\ell \le H \le \tilde{H}$ for $i \in J_\ell$ and $H^{(i)}_s \le H \le \tilde{H}$ for $i \in J_s$.*

Proof. If $i \in J_\ell$ and $\alpha \in \dot{R}_{lg}$, then $\alpha + k_i\delta_i = \alpha + \delta_i \in R_{lg}$. Thus $t_\alpha^{(i)} = r_{\alpha+\delta_i} r_\alpha \in H$. From Lemma 3.11(i) we get $t_\alpha^{(i)} \in H$ for all $\alpha \in \dot{Q}_\ell$ and therefore $H_\ell^{(i)} \leq H \leq \tilde{H}$. If $i \in J_s$ and $\alpha \in \dot{R}_{sh}$, then $\alpha + \delta_i \in R_{sh}$. Thus $t_{k\alpha}^{(i)} = r_{\alpha+\delta_i} r_\alpha \in H$. So by Lemma 3.11(i), $t_{k\alpha}^{(i)} \in H$ for all $\alpha \in \dot{Q}_s$. This gives $H_s^{(i)} \leq H \leq \tilde{H}$. □

Remark 3.20 *If $\nu = 1$, then we are in the case of the theory of affine Weyl groups which is known. So for the sake of simplicity we assume from now on that $\nu > 1$. However as is known (see [Ka] for example), Proposition 3.30 of this section is valid for the case $\nu = 1$. So we will refer to Proposition 3.30, in future sections, for $\nu \geq 1$.*

Let

$$C = \langle c_{ij} : i, j \in J \rangle \leq Aut(\tilde{\mathcal{V}}),$$

$$\tilde{\mathcal{Z}} = C \cap \tilde{H} \quad \text{and}$$

$$\mathcal{Z} = \tilde{\mathcal{Z}} \cap H = C \cap H.$$

From (2.7) we have that $(\alpha, \alpha) = 2k$ for $\alpha \in \dot{R}_{lg}$ then it follows that $(\alpha, \beta) \in k\mathbf{Z}$ for $\alpha, \beta \in Q_\ell$. Because of this the part (iv) of the following lemma makes sense.

Lemma 3.21 *(i) C, $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$ are free abelian groups of rank $\nu(\nu-1)/2$.*

(ii) For any $T \in GL(\tilde{\mathcal{V}})$ such that T fixes pointwise $\mathcal{V}^0$ and the bilinear form $(\cdot,\cdot)$ is invariant under T we have $c_{ij}T = Tc_{ij}$, $i, j \in J$.

(iii) $wc_{ij} = c_{ij}w$ and $c_{ij}t_\alpha^{(n)} = t_\alpha^{(n)}c_{ij}$, for any $w \in \tilde{\mathcal{W}}$, $i, j, n \in J$ and $\alpha \in Q$,

(iv) $(t_\alpha^{(i)}, t_\beta^{(j)}) = c_{ij}^{\frac{1}{k}(\alpha,\beta)}$, $\alpha, \beta \in Q_\ell$, $i, j \in J$, where $(x, y) = xyx^{-1}y^{-1}$ is the group commutator.

Proof. First it is easy to see that C is an abelian subgroup of $Aut(\tilde{\mathcal{V}})$. By (3.18), C is generated by $\nu(\nu-1)/2$ elements $\{c_{ij} : i, j \in J, i < j\}$. Suppose

$$\prod_{i<j} c_{ij}^{n_{ij}} = 1$$

for some integers n_{ij}. Then for $\lambda \in \tilde{\mathcal{V}}$ we have,

$$\lambda = \prod_{i<j} c_{ij}^{n_{ij}} \lambda = \lambda + \sum_{i<j} \frac{1}{k} n_{ij}((\lambda, \delta_i)\delta_j - (\lambda, \delta_j)\delta_i).$$

Thus

$$\sum_{i<j} n_{ij}((\lambda, \delta_i)\delta_j - (\lambda, \delta_j)\delta_i) = 0.$$

Subsituting $\lambda = \Lambda_k$, $k \in J$, we get $n_{ij} = 0$ for all $i, j \in J$, $i < j$. So C is free. To show $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$ are free abelian goups of rank $\nu(\nu-1)/2$ it is enough to show that for any $i, j \in J$ there exist some non-zero integers n_{ij} such that $c_{ij}^{n_{ij}} \in \mathcal{Z}$, this is because any non-trivial subgroup of a free abelian group is free and $\mathcal{Z} \subseteq \tilde{\mathcal{Z}} \subseteq C$. Fix $i, j \in J$ and $\dot{\alpha} \in \dot{R}_{sh}$. We have $\dot{\alpha} + S + kS \subseteq \dot{\alpha} + S \subseteq R_{sh}$. Therefore $\dot{\alpha} + \delta_i$, $\dot{\alpha} - k\delta_j$ and $\dot{\alpha} + \delta_i - k\delta_j$ are in R_{sh}. Now by Lemma 3.12(ii), we have

$$r_{\dot{\alpha}+\delta_i-k\delta_j} r_{\dot{\alpha}-k\delta_j} = (t^{(i)}_{k(\dot{\alpha}-k\delta_j)}) = t^{(i)}_{k\dot{\alpha}} t^{(i)}_{-k^2\delta_j} = t^{(i)}_{k\dot{\alpha}} c_{ij}^{k^2}.$$

We have $(t^{(i)}_{k\dot{\alpha}}) = r_{\dot{\alpha}+\delta_i} r_{\dot{\alpha}} \in H$ and $r_{\dot{\alpha}+\delta_i-k\delta_j} r_{\dot{\alpha}-k\delta_j} \in H$. Therefore

$$c_{ij}^{k^2} = t^{(i)}_{k\dot{\alpha}} r_{\dot{\alpha}-k\delta_j+\delta_i} r_{\dot{\alpha}-k\delta_j} \in H \cap C = \mathcal{Z}.$$

This completes the proof of part (i).

(ii) By Lemma 3.11(ii), we have

$$T c_{ij} T^{-1} = T t^{(i)}_{-\delta_j} T^{-1} = t^{(i)}_{T(-\delta_j)} = t^{(i)}_{-\delta_j} = c_{ij}.$$

(iii) This follows immediately from (ii).

(iv) Using Lemma 3.11 and (3.10) we have

$$\begin{aligned}
(t^{(i)}_{\alpha}, t^{(j)}_{\beta}) &= t^{(i)}_{\alpha} t^{(j)}_{\beta} t^{(i)}_{-\alpha} t^{(j)}_{-\beta} \\
&= t^{(j)}_{t^{(i)}_{\alpha}(\beta)} t^{(j)}_{-\beta} = t^{(j)}_{t^{(i)}_{\alpha}(\beta)-\beta} \\
&= t^{(j)}_{\frac{1}{k}(\beta,\alpha)\delta_i} = t^{(i)}_{-\frac{1}{k}(\beta,\alpha)\delta_j} \\
&= c_{ij}^{\frac{1}{k}(\alpha,\beta)}.
\end{aligned}$$

□

From part (iii) of Lemma 3.21 we have

Corollary 3.22 *$\tilde{\mathcal{Z}} \subseteq Cent(\tilde{\mathcal{W}})$ and $\mathcal{Z} \subseteq Cent(\mathcal{W})$.*

Corollary 3.23 *(i)* $\tilde{H} = \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$.

(ii) $H = \mathcal{Z}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$

Proof. (i) We have $\tilde{\mathcal{Z}} \subseteq \tilde{H}$ and by Lemma 3.19, $H_s^{(i)} \subseteq \tilde{H}$ for $i \in J_s$ and $H_\ell^{(j)} \subseteq \tilde{H}$ for $j \in J_\ell$. Thus the right hand side of the equality in (i) is a subset of $\tilde{H}$. To show $\tilde{H}$ is a subset of the right hand side, we only need to show that $t_\alpha^{(i)} \in \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$ for all $(i,\alpha) \in (J_\ell \times Q_\ell) \cup (J_s \times Q_s)$, this is because of Lemma 3.15. First let $(i,\alpha) \in J_\ell \times R_{lg}$. Then $\alpha = \dot{\alpha} + \delta$, where $\delta = \sum_{i=1}^{\nu} m_i\delta_i$, $\dot{\alpha} \in \dot{R}_{lg}$ and $m_i \in \mathbf{Z}$. So

$$t_\alpha^{(i)} = t_{\dot{\alpha}}^{(i)} t_\delta^{(i)} = t_{\dot{\alpha}}^{(i)} (t_{\delta_\nu}^{(i)})^{m_\nu} (t_{\delta_{\nu-1}}^{(i)})^{m_{\nu-1}} \cdots (t_{\delta_1}^{(i)})^{m_1} \quad = t_{\dot{\alpha}}^{(i)} z$$

for some $z \in C$. Therefore $z = t_{-\dot{\alpha}}^{(i)} t_\alpha^{(i)} \in \tilde{H} \cap C = \tilde{\mathcal{Z}}$. Thus

$$t_\alpha^{(i)} \in \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)}).$$

Then from Lemma 3.11(i) we get that $t_\alpha^{(i)} \in \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$ for any $(i,\alpha) \in J_\ell \times Q_\ell$. Now let $(i, k\alpha) \in J \times kR_{sh}$ where $\alpha = \dot{\alpha} + \delta$, $\dot{\alpha} \in \dot{R}_{sh}$ and $\delta = \sum_{i=1}^{\nu} m_i\delta_i$, $m_i \in \mathbf{Z}$. Then as in the first part of (i) we get

$$t_{k\alpha}^{(i)} = t_{k\dot{\alpha}+k\delta}^{(i)} = t_{k\dot{\alpha}}^{(i)} t_{k\delta}^{(i)} = t_{k\dot{\alpha}}^{(i)} z$$

for some $z \in C$. Then $z \in \tilde{H} \cap C = \tilde{\mathcal{Z}}$. Thus $t_{k\alpha}^{(i)} \in \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$. Again using Lemma 3.11(i) we get $t_{k\alpha}^{(i)} \in \tilde{\mathcal{Z}}(\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$ for any $(i,\alpha) \in J_s \times Q_s$.

(ii) By Lemma 3.19, the right hand side is a subset of H. Note let $h \in H$. Since $H \subseteq \tilde{H}$, then by part (i), there are elements $z \in C$ and $h' \in (\prod_{i\in J_s} H_s^{(i)})(\prod_{j\in J_\ell} H_\ell^{(j)})$ such that $h = zh'$. We are done if we show that $z \in \mathcal{Z}$. But $h' \in H$, so $z = hh'^{-1} \in C \cap H = \mathcal{Z}$. This completes the proof. □

By the above corollary, any element $h \in \tilde{H}$ can be written in the form

$$h = z(\prod_{i\in J_s} t_{k\alpha_i}^{(i)})(\prod_{j\in J_\ell} t_{\beta_j}^{(j)}),\ (\alpha_i \in \dot{Q}_s, \beta_j \in \dot{Q}_\ell,\ z \in \tilde{\mathcal{Z}}). \tag{3.24}$$

Corollary 3.25 *Let $h \in \tilde{H}$ be written in the form (3.24). Then α_i's and β_j's are independent of the choice of the order of the product. In particular if we take the natural ordering on J_s and J_ℓ, then the expression of $h \in \tilde{H}$ in the form (3.24) is unique.*

Proof. Let $p \in J$, $j \in J_\ell$ and $i \in J_s$. Then for $\beta \in \dot{Q}_\ell$,

$$t_\beta^{(j)} \Lambda_p \equiv \Lambda_p - \frac{1}{k}\delta_{pj}\beta, \quad (mod\mathcal{V}^0),$$

and for $\alpha \in \dot{Q}_s$,

$$t_{k\alpha}^{(i)} \Lambda_p \equiv \Lambda_p - \delta_{pj}\alpha \quad (mod\mathcal{V}^0).$$

If $p \in J_\ell$, then considering (3.10) and (3.18) we have

$$h\Lambda_p \equiv zt_{\beta_p}^{(p)} \Lambda_p \equiv \Lambda_p - \frac{1}{k}\beta_p, \quad (mod\mathcal{V}^0),$$

and if $p \in J_s$, then

$$h\Lambda_p \equiv zt_{k\alpha_p}^{(p)} \Lambda_p \equiv \Lambda_p - \alpha_p, \quad (mod\mathcal{V}^0).\square$$

Remark 3.26 *Note that Corollary 3.25 is valid if $\tilde{H}$ and $\tilde{\mathcal{Z}}$ are replaced with H and $\mathcal{Z}$, respectively.*

Recall that if G is a group with $(G,G) \subseteq Cent(G)$, when $(a,b) = aba^{-1}b^{-1}$ is the group commutator, then

$$(\prod_{i \in I} x_i, y) = \prod_{i \in I}(x_i, y) \quad \text{for } x_i, y \in G, \ I \text{ a finite set.} \tag{3.27}$$

Proposition 3.28 *(i) $\tilde{\mathcal{Z}} = Cent(\tilde{H})$ and $\mathcal{Z} = Cent(H)$,*

(ii) $(\tilde{H}, \tilde{H}) = (H, H) \subseteq \mathcal{Z}$,

(iii) Both $\frac{H}{\mathcal{Z}}$ and $\frac{\tilde{H}}{\tilde{\mathcal{Z}}}$ are canonically isomorphic to the direct product of t copies of $k\dot{Q}_s$ and $(\nu - t)$ copies of $\dot{Q}_\ell$.

Proof. (i) From Corollary 3.22 we have $\tilde{\mathcal{Z}} \subseteq Cent(\tilde{H})$. Now let $h \in Cent(\tilde{H}) \subseteq \tilde{H}$. Let $h = z(\prod_{i \in J_s} t_{k\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{\beta_j}^{(j)})$ be an expression of h in the form (3.24). We have $(h, t_{k\alpha}^{(p)}) = 1$ for all $(p, \alpha) \in J_s \times Q_s$, and $(h, t_\beta^{(p)}) = 1$ for all $(p, \beta) \in J_\ell \times Q_\ell$. Thus for $(p, \alpha) \in J_\ell \times Q_\ell$ we have, using 3.27,

$$\begin{aligned} 1 = (h, t_\alpha^{(p)}) &= ((\prod_{i \in J_s} t_{k\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{\beta_j}^{(j)}), t_\alpha^{(p)}) \\ &= \prod_{i \in J_s} (t_{k\alpha_i}^{(i)}, t_\alpha^{(p)}) \prod_{j \in J_\ell} (t_{\beta_j}^{(j)}, t_\alpha^{(p)}) \\ &= \prod_{i \in J_s} c_{ip}^{(\alpha_i, \alpha)} \prod_{j \in J_\ell} c_{jp}^{\frac{1}{k}(\beta_j, \alpha)}. \end{aligned}$$

where the last equality follows from Lemma 3.21. Now using Lemma 3.21(i) we get that $(\alpha_i,\alpha)=0$ for all $i\in J_s$, $\alpha\in Q_\ell$ and $(\beta_j,\alpha)=0$ for all $j\in J_\ell$, $\alpha\in Q_\ell$. Since $span_{\mathbb{R}}\dot{Q}_\ell=\dot{\mathcal{V}}$ and the restriction of the form on $\dot{\mathcal{V}}$ is positive definite we get that $\alpha_i=0$ for all $i\in J_s$ and $\beta_j=0$ for all $j\in J_\ell$. Thus $h\in\tilde{\mathcal{Z}}$ and $Cent(\tilde{H})=\tilde{\mathcal{Z}}$. We can use a similar argument to see that $Cent(H)=\mathcal{Z}$ (see Remark 3.26).

(ii) From Corollary 3.23 and part (i) we have

$$\begin{aligned}
(\tilde{H},\tilde{H}) &= (\tilde{\mathcal{Z}}(\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}),\tilde{\mathcal{Z}}(\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}))\\
&= ((\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}),(\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}))\\
&= (\mathcal{Z}(\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}),\mathcal{Z}(\prod_{i\in J_s}H_s^{(i)})(\prod_{i\in J_\ell}H_\ell^{(i)}))\\
&= (H,H)\subseteq\mathcal{Z},
\end{aligned}$$

where the last inclusion follows from Lemma 3.21(iv).

(iii) We consider the surjective group homomorphisms

$$\tilde{H}\to k\dot{Q}_s\times\cdots\times k\dot{Q}_s\times\dot{Q}_\ell\times\cdots\times\dot{Q}_\ell\quad(t\text{-times } k\dot{Q}_s\quad\text{and}\quad(\nu-t)\text{-times }\dot{Q}_\ell),$$

$$H\to k\dot{Q}_s\times\cdots\times k\dot{Q}_s\times\dot{Q}_\ell\times\cdots\times\dot{Q}_\ell\quad(t\text{-times } k\dot{Q}_s\quad\text{and}\quad(\nu-t)\text{-times }\dot{Q}_\ell),$$

$$h=z(\prod_{i\in J_s}t^{(i)}_{k\alpha_i})(\prod_{j\in J_\ell}t^{(j)}_{\beta_j})\mapsto(k\alpha_1,\ldots,k\alpha_t,\beta_{t+1},\ldots,\beta_\nu).$$

Note that by Corollary 3.25, the above maps are well defined. It is clear that the kernel of these homomorphisms are $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$, respectively. Thus both sequences

$$1\to\tilde{\mathcal{Z}}\to\tilde{H}\to k\dot{Q}_s\times\cdots\times k\dot{Q}_s\times\dot{Q}_\ell\times\cdots\times\dot{Q}_\ell\to 1$$

$$1\to\mathcal{Z}\to H\to k\dot{Q}_s\times\cdots\times k\dot{Q}_s\times\dot{Q}_\ell\times\cdots\times\dot{Q}_\ell\to 1$$

are exact, which gives (iii). □

The following example shows that the inclusions in part (ii) of Proposition 3.37 can be proper.

Example 3.29 *Let $\dot{R}$ be a finite root system of type A_1(including 0). Then by Theorem I.1.28,*

$$R=\dot{R}\oplus\mathbf{Z}\delta_1\oplus\mathbf{Z}\delta_2$$

is an extended affine root system of type A_1. Using Lemma 3.21(iv), we have for any $\alpha, \beta \in R^\times$, $(t_\alpha^{(i)}, t_\beta^{(j)}) = c_{ij}^{(\alpha,\beta)}$, $1 \le i, j \le 2$. Since $(\alpha, \beta) \in 2\mathbf{Z}$ for any $\alpha, \beta \in R$, we get

$$c_{ij} \notin (H, H).$$

On the other hand if $\alpha \in R^\times$, then $\alpha+\delta_i+\delta_j$ and $\alpha+\delta_i$ are in $R^\times$, $i < j$ and $r_{\alpha+\delta_i+\delta_j} r_{\alpha+\delta_i} = t_{\alpha+\delta_j}^{(i)} = t_\alpha^{(i)} c_{ij}^{-1}$. Therefore $c_{ij} = t_{-\alpha-\delta_j}^{(i)} t_\alpha^{(i)} \in H \cap C = \mathcal{Z}$. Thus

$$c_{ij} \in \mathcal{Z} \setminus (H, H).$$

Proposition 3.30 *$\tilde{\mathcal{W}} = \dot{\mathcal{W}} \propto \tilde{H}$ and $\mathcal{W} = \dot{\mathcal{W}} \propto H$.*

Proof. Each generator of $\tilde{\mathcal{W}}$ has the form $r_{\dot\alpha+\delta}$ where $\dot\alpha \in \dot{R}^\times$, $\delta \in \mathcal{V}^0$ and $\dot\alpha + \delta \in \tilde{R}$. Since $r_{\dot\alpha+\delta} r_{\dot\alpha} \in \tilde{H}$, we have $\tilde{\mathcal{W}} = \langle \dot{\mathcal{W}}, \tilde{H} \rangle$. A similar argument shows that $\mathcal{W} = \langle dot\mathcal{W}, H \rangle$. By 3.14, $\tilde{H} \lhd \tilde{\mathcal{W}}$ and $H \lhd \mathcal{W}$. We need to show that $\tilde{H} \cap \dot{\mathcal{W}} = 1$ and $H \cap \dot{\mathcal{W}} = 1$. But $H \subseteq \tilde{H}$,so we only need to show that $\tilde{H} \cap \dot{\mathcal{W}} = 1$. For this it is enough to show that $\tilde{H}$ is a torsion free subgroup of $\tilde{\mathcal{W}}$, since $\dot{\mathcal{W}}$ is a finite subgroup of $\tilde{\mathcal{W}}$. Let $h \in \tilde{H}$, $h \neq 1$ and $h^n = 1$ for some $n \in \mathbf{Z}$. We want to show $n = 0$. Let $h = z(\prod_{i \in J_s} t_{k\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{\beta_j}^{(j)})$ be an expression of h in the form (3.24). If $\alpha_i = \beta_j = 0$ for all $i \in J_s$ and $j \in J_\ell$, then $h^n = z^n$. Since $\tilde{\mathcal{Z}}$ is a free abelian group we get $n = 0$. Now assume there is some $i_0 \in J_s$ so that $\alpha_{i_0} \neq 0$. By Lemma 3.21(iv), and Proposition 3.28,

$$h^n = (\prod_{i \in J_s \setminus \{i_0\}} t_{nk\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{n\beta_j}^{(j)}) t_{nk\alpha_{i_0}}^{(i_0)} z' \quad \text{for some } z' \in \tilde{\mathcal{Z}}.$$

Since the restriction of the form to $\dot{Q}$ is positive definite we have $(\alpha_{i_0}, \alpha_{i_0}) \neq 0$. Using (3.10) and the fact that the elements of $\tilde{\mathcal{Z}}$ act as the identity on $\dot{Q}$ we get

$$\begin{aligned} h^n(\alpha_{i_0}) &= (\prod_{i \in J_s \setminus \{i_0\}} t_{nk\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{n\beta_j}^{(j)}) t_{nk\alpha_{i_0}}^{(i_0)}(\alpha_{i_0}) \\ &= (\prod_{i \in J_s \setminus \{i_0\}} t_{nk\alpha_i}^{(i)})(\prod_{j \in J_\ell} t_{n\beta_j}^{(j)})(\alpha_{i_0}) + (\alpha_{i_0}, n\alpha_{i_0})\delta_{i_0} \\ &\equiv (\alpha_{i_0}, n\alpha_{i_0})\delta_{i_0} \quad (mod\ \dot{Q} + \sum_{i \in J \setminus \{i_0\}} \mathbf{Z}\delta_i). \end{aligned}$$

So if $n \neq 0$, then $h^n(\alpha_{i_0}) \neq \alpha_{i_0}$ which is a contradiction. Thus $n = 0$. If there is some $j \in J_\ell$ so that $\beta_j \neq 0$, then using an analogous arguement as above we get $n = 0$. Thus $\tilde{H}$ is torsion free. □

We remark here that to prove $\tilde{H}$ is torsion free in the above proposition, we could use Corollary 3.25, making the argument simpler. But for a later purpose we avoided doing that.

Recall that for a group G and $x_0 \in G$, the subgroup $C_G(x_0) = \{x \in G \mid xx_0 = x_0x\}$ is called the centralizer of x_0 in G. For a subgroup N of G, the subgroup $C_G(N) = \{x \in G \mid xn = nx, \text{for all } n \in N\}$ is called the centralizer of N in G. N is called self-centralizing if $C_G(N) = N$. Also the subgroup $G_{fc} = \{x \in G \mid [G : C_G(x)] < \infty\}$ of G is called the finite conjugacy subgroup of G. In fact G_{fc} is a characteristic subgroup of G; that is, it is invariant under any automorphism of G.

Lemma 3.31 *(i)* $C_{\mathcal{W}/\mathcal{Z}}(H/\mathcal{Z}) = H/\mathcal{Z}$ *and* $C_{\tilde{\mathcal{W}}/\tilde{\mathcal{Z}}}(\tilde{H}/\tilde{\mathcal{Z}}) = \tilde{H}/\tilde{\mathcal{Z}}$.
(ii) $C_{\mathcal{W}}(H) = \mathcal{Z}$ *and* $C_{\tilde{\mathcal{W}}}(\tilde{H}) = \tilde{\mathcal{Z}}$.

Proof. Let $w \in \mathcal{W}$ and $w\mathcal{Z} \in C_{\mathcal{W}/\mathcal{Z}}(H/\mathcal{Z})$. Then $w^{-1}h^{-1}wh \in \mathcal{Z}$ for all $h \in H$. In particular $t^{(i)}_{\alpha-w^{-1}\alpha} = w^{-1}t^{(i)}_{-\alpha}wt^{(i)}_{\alpha} \in \mathcal{Z}$ for all $(i,\alpha) \in (J_\ell \times \dot{Q}_\ell) \cup (J_s \times k\dot{Q}_s)$. From Corollary 3.25 we get that $\alpha - w^{-1}\alpha \equiv 0\ (mod\mathcal{V}^0)$. By Proposition 3.30, we have $w = \dot{w}h$ for some $\dot{w} \in \dot{\mathcal{W}}$ and $h \in H$. Therefore $\alpha \equiv w^{-1}\alpha \equiv h^{-1}\dot{w}^{-1}\alpha\ (mod\mathcal{V}^0)$. Since $\mathcal{W}$ fixes pointwise $\mathcal{V}^0$, we get $h\alpha \equiv \dot{w}^{-1}\alpha\ (mod\mathcal{V}^0)$. Now we have $\dot{\mathcal{W}}\dot{Q} \subseteq \dot{Q}$ and by (3.10), $h\alpha \equiv \alpha\ (mod\mathcal{V}^0)$ for any $\alpha \in \dot{Q}$. So $\dot{w}^{-1}\alpha = \alpha$ for all $\alpha \in \dot{Q}$. Thus $\dot{w} = 1$ (see [H]) and so $w = h \in H$. So $C_{\mathcal{W}/\mathcal{Z}}(H/\mathcal{Z}) \subseteq H/\mathcal{Z}$. Since $H/\mathcal{Z}$ is abelian we get $C_{\mathcal{W}/\mathcal{Z}}(H/\mathcal{Z}) = H/\mathcal{Z}$. An analogous argument gives $C_{\tilde{\mathcal{W}}/\tilde{\mathcal{Z}}}(\tilde{H}/\tilde{\mathcal{Z}}) = \tilde{H}/\tilde{\mathcal{Z}}$.

(ii) From part (i) we have $C_{\mathcal{W}}(H) \subseteq H$. Then from Lemma 3.28(i) we get $C_{\mathcal{W}}(H) = \mathcal{Z}$. Similarly $C_{\tilde{\mathcal{W}}}(\tilde{H}) = \tilde{\mathcal{Z}}$. A similar argument gives $\tilde{\mathcal{Z}} = Cent(\tilde{\mathcal{W}})$. □

Corollary 3.32 $\mathcal{Z} = Cent(\mathcal{W})$ *and* $\tilde{\mathcal{Z}} = Cent(\tilde{\mathcal{W}})$.

Proof. By Lemma 3.31, $Cent(\mathcal{W})/\mathcal{Z} \subseteq C_{\mathcal{W}/\mathcal{Z}}(H/\mathcal{Z}) \subseteq H/\mathcal{Z}$. Thus $Cent(\mathcal{W}) \subseteq H$ and so $Cent(\mathcal{W}) \subseteq Cent(H) = \mathcal{Z}$. Then by Corollary 3.22, we get $\mathcal{Z} = Cent(\mathcal{W})$. □

We recall the following lemma from [P].

Lemma 3.33 *Let G be a group, N be a normal torsion free abelian subgroup of G with* $]G : N] < \infty$. *Then $G_{fc} = N$ if and only if N is self-centralizing.*

Theorem 3.34 *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two EAWGs of the form under consideration and φ : $\mathcal{W}_1 - \mathcal{W}_2$ be a group isomorphism. Let $\dot{\mathcal{W}}_1$, $\dot{\mathcal{W}}_2$, H_1 and H_2 be subgroups of $\mathcal{W}_1$ and $\mathcal{W}_2$ defined by (3.13) and (3.4), respectively. Then $\dot{\mathcal{W}}_1 \cong \dot{\mathcal{W}}_2$ and $\varphi(H_1) = H_2$.*

Proof. We denote by $\mathcal{Z}_1$ and $\mathcal{Z}_2$ the centers of $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. From Corollary 3.32 we know that $\mathcal{Z}_1 = Cent(H_1)$ and $\mathcal{Z}_2 = Cent(H_2)$ and by Proposition 3.30, $\mathcal{W}_1 = \dot{\mathcal{W}}_1 \propto H_1$ and $\mathcal{W}_2 = \dot{\mathcal{W}}_2 \propto H_2$. Since φ is an isomorphism we have

$$\dot{\mathcal{W}}_2 \propto H_2 = \mathcal{W}_2 = \varphi(\mathcal{W}_1) = \varphi(\dot{\mathcal{W}}_1) \propto \varphi(H_1).$$

Since $\mathcal{Z}_i = Cent(\mathcal{W}_i)$ for $i = 1,2$, therefore $\dot{\mathcal{W}}_i\mathcal{Z}_i = \{\dot{w}z : \dot{w} \in \dot{\mathcal{W}}_i, z \in \mathcal{Z}_i\}$, $i = 1,2$ is a subgroup of $\mathcal{W}_i$. We have

$$\frac{\dot{\mathcal{W}}_2\mathcal{Z}_2}{\mathcal{Z}_2} \propto \frac{H_2}{\mathcal{Z}_2} = \frac{\mathcal{W}_2}{\mathcal{Z}_2} = \frac{\varphi(\mathcal{W}_1)}{\varphi(\mathcal{Z}_1)} = \frac{\varphi(\dot{\mathcal{W}}_1)\cdot\varphi(\mathcal{Z}_1)}{\varphi(\mathcal{Z}_1)} \propto \frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)}.$$

Since φ is an isomorphism and because of Proposition 3.28(iii) we have that $\frac{H_2}{\mathcal{Z}_2}$ and $\frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)}$ are free abelian groups. Also from Lemma 3.31 we have

$$C_{\frac{\mathcal{W}_2}{\mathcal{Z}_2}}(\frac{H_2}{\mathcal{Z}_2}) \subseteq \frac{H_2}{\mathcal{Z}_2} \quad \text{and} \quad C_{\frac{\mathcal{W}_2}{\mathcal{Z}_2}}(\frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)}) = C_{\frac{\varphi(\mathcal{W}_1)}{\varphi(\mathcal{Z}_1)}}(\frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)}) \subseteq \frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)}.$$

Moreover

$$[\frac{\mathcal{W}_2}{\mathcal{Z}_2}, \frac{H_2}{\mathcal{Z}_2}] = |\dot{\mathcal{W}}_2| < \infty \quad \text{and} \quad [\frac{\mathcal{W}_2}{\mathcal{Z}_2}, \frac{\varphi(H_1)}{\varphi(Z_1)}] = |\varphi(\dot{\mathcal{W}}_1)| < \infty.$$

Therefore by Lemma 3.33, we have

$$\frac{H_2}{\mathcal{Z}_2} = (\frac{\mathcal{W}_2}{\mathcal{Z}_2})_{fc} = \frac{\varphi(H_1)}{\varphi(\mathcal{Z}_1)} = \frac{\varphi(H_1)}{\mathcal{Z}_2}.$$

Thus $\varphi(H_1) = H_2$. Since $\mathcal{W}_2 = \varphi(\mathcal{W}_1) = \varphi(\dot{\mathcal{W}}_1) \propto \varphi(H_1) = \varphi(\dot{\mathcal{W}}_1) \propto H_2$ we have

$$\dot{\mathcal{W}}_2 \cong \tfrac{\mathcal{W}_2}{H_2} \cong \varphi(\dot{\mathcal{W}}_1) \cong \dot{\mathcal{W}}_1.$$

□

Corollary 3.35 *H is a characteristic subgroup of $\mathcal{W}$.* □

Let

$$R_1 = (S_1 + S_1) \cup (\dot{R}_{sh} + S_1) \cup (\dot{R}_{lg} + L_1) \quad \text{and}$$

$$R_2 = (S_2 + S_2) \cup (\dot{R}_{sh} + S_2) \cup (\dot{R}_{lg} + L_2)$$

be reduced EARS's of the form as in Convention 2.3. Furthermore assume that

$$\langle S_1 \rangle = \langle S_2 \rangle \quad \text{and} \quad \langle L_1 \rangle = \langle L_2 \rangle.$$

Then we have

Proposition 3.36 $\mathcal{W}_1 = \mathcal{W}_2 \Leftrightarrow Cent(\mathcal{W}_1) = Cent(\mathcal{W}_2)$.

Proof. We denote by H_n and $\mathcal{Z}_n$ the subgroups H and $\mathcal{Z}$ of $\mathcal{W}_n$, for $n = 1, 2$. From Proposition 3.30 and Corollaries 3.23 and 3.32 we have

$$\mathcal{W}_n = \dot{\mathcal{W}} \propto H_n, \quad H_n = \mathcal{Z}_n(\prod_{i \in J_s} H_s^{(i)})(\prod_{j \in J_\ell} H_\ell^{(j)}) \quad \text{and} \quad \mathcal{Z}_n = Cent(\mathcal{W}_n).$$

for $n = 1, 2$. Now the result is clear. □

It is well known that nonisomorphic finite or affine root systems might have isomorphic Weyl groups. The following example shows that for arbitrary ν, nonisomorphic EARS's might have isomorphic EAWG's. Unlike the finite or affine case even when the type is the same, nonisomorphic EARS's might have isomorphic EAWG's, as part (ii) of the following example shows.

Example 3.37 *(i) Let ν be arbitrary and R be an EARS of type $B_\ell (l \geq 3)$. By Proposition 2.9, $\check{R}$ is an EARS of type $C_\ell (l \geq 3)$. So $R \not\cong \check{R}$. From (2.8) we have $\check{R}^\times = \{\check{\alpha} : \alpha \in R^\times\}$. So*

$$\mathcal{W}_{\check{R}} = \langle r_{\check{\alpha}} : \check{\alpha} \in \check{R}^\times \rangle = \langle r_\alpha : \alpha \in R^\times \rangle = \mathcal{W}_R.$$

(ii) Let R ba an extended affine root system of type A_1 with nullity $\nu \geq 3$. By Theorem I.1.28, R has the form $R = (S + S) \cup (\dot{R} + S)$, where S is a semilattice and $\dot{R}$ is a finite root system of type A_1. As in (3.1) we can write $\langle S \rangle = \bigoplus_{i=1}^{\nu} \mathbf{Z}\delta_i$ where $\delta_i \in S$. Set

$$S_1 := \cup_{i,j=1}^{\nu} (\mathbf{Z}\delta_i + \mathbf{Z}\delta_j + 2\langle S \rangle).$$

S_1 is a semilattice, since it is a union of cosets of $2\langle S \rangle$ in S including the trivial coset $2\langle S \rangle$ (see Remark 1.17). Moreover $\langle S_1 \rangle = \langle S \rangle$. Set

$$R_1 := (S_1 + S_1) \cup (\dot{R} + S_1).$$

By Theorem I.1.28, R_1 is also an extended affine root system of type A_1. We show $\mathcal{W}_{R_1} = \tilde{\mathcal{W}}$. By Proposition 3.36, we only need to show that $\tilde{\mathcal{Z}} = \mathcal{Z}_1$ where $\mathcal{Z}_1$ is the center of $\mathcal{W}_{R_1}$. Since $\mathcal{Z}_1 \subseteq \tilde{\mathcal{Z}} \subseteq C = \langle c_{ij} : i, j \in J \rangle$, we get the result if we prove that $c_{ij} \in \mathcal{Z}_1$, for all $i, j \in J$. Let $\dot{\alpha} \in \dot{R}$. If we denote the corresponding subgroup H of R_1 by H_1, then

$$c_{ij} t_{\dot{\alpha}}^{(i)} = t_{\dot{\alpha}-\delta_j}^{(i)} = r_{\dot{\alpha}-\delta_j+\delta_i} r_{\dot{\alpha}-\delta_j} \in H_1.$$

Therefore $c_{ij} = r_{\dot{\alpha}-\delta_j+\delta_i} r_{\dot{\alpha}-\delta_j} t_{\dot{\alpha}}^{(i)} \in \mathcal{W}_{R_1}$. Thus $c_{ij} \in \mathcal{Z}_1$ and so $\mathcal{W}_{R_1} = \tilde{\mathcal{W}}$. Since $\langle S \rangle$ is a lattice and S_1 is not a lattice, R_1 and $\tilde{R}$ are not isomorphic (see Theorem I.1.29).

We close this section with the follwing remark.

Remark 3.38 *(i) Let $0 \leq \mu \leq \nu$ and let J_μ be a subset of J of cardinality μ. Similar to the definition of $\tilde{\mathcal{V}}$ in (3.3) we define*

$$\tilde{\mathcal{V}}_\mu := \sum_{i=1}^{l} \mathbf{R}\dot{\alpha}_i \oplus \sum_{i \in J} \mathbf{R}\delta_i \oplus \sum_{i \in J_\mu} \mathbf{R}\Lambda_i.$$

As in (2.5) we extend the bilinear form $(\cdot,\cdot)$ on $\mathcal{V}$ to $\tilde{\mathcal{V}}_\mu$ by

- *$(\cdot,\cdot)$ extends the form $(\cdot,\cdot)$ on $\mathcal{V}$.*
- *$(\dot{\mathcal{V}}, \sum_{i\in J_\mu} \mathbf{R}\Lambda_i) = \{0\}$, $\quad (\sum_{i\in J_\mu} \mathbf{R}\Lambda_i, \sum_{i\in J_\mu} \mathbf{R}\Lambda_i) = \{0\}$,*
- *$(\delta_i, \Lambda_j) = \delta_{ij}$ for $i \in J$ and $j \in J_\mu$.*

Thus the form has a $(\nu - \mu)-$dimensional radical. For $\alpha \in \tilde{\mathcal{V}}_\mu$, nonisotropic, we define

$$r_\alpha(\lambda) = \lambda - (\lambda, \check{\alpha})\alpha, \quad (\lambda \in \tilde{\mathcal{V}}_\mu).$$

Set

$$\mathcal{W}_\mu = \langle r_\alpha : \alpha \in R^\times \rangle \leq GL(\tilde{\mathcal{V}}_\mu).$$

Define $\tilde{H}_\mu$, H_μ, C_μ, $\tilde{\mathcal{Z}}_\mu$ and $\mathcal{Z}_\mu$ exactly in the same way we defined $\tilde{H}$, H, C, $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$, respectively. Checking the proofs in this section, we see that the only places that $|J_\mu|$ plays a role are in Lemma 3.21(i) and Corollary 3.25. Note that we deliberately did not use Corollary 3.25 to prove any other result in this section. Also one can see that Lemma 3.21(i) holds for C_μ, $\tilde{\mathcal{Z}}_\mu$ and $\mathcal{Z}_\mu$ by changing the statement to the following:

$$C_\mu, \tilde{\mathcal{Z}}_\mu \text{ and } \mathcal{Z}_\mu \text{ are free abelian groups of rank } \frac{\mu(\mu-1)}{2} + \mu(\nu - \mu).$$

Therefore all the results of this section, with the exceptions mentioned above, remain true for $\tilde{\mathcal{W}}_\mu$, $\mathcal{W}_\mu$, $\tilde{\mathcal{Z}}_\mu$ *and* $\mathcal{Z}_\mu$.

(ii) $\mathcal{W}_\mu$ *is a central extension of* $\mathcal{W}_0$. *To see this, consider the sequence*

$$1 \to \mathcal{Z}_\mu \xrightarrow{id} \mathcal{W}_\mu \xrightarrow{\pi_\mu} \mathcal{W}_0 \to 1$$

where π_μ *is the surjective homomorphism defined by* $\pi_\mu(w) = w_{|\mathcal{V}}$. *We need to show that* $ker\pi_\mu = \mathcal{Z}_\mu$. *Since elements of* $\mathcal{Z}_\mu$ *act as the identity on* $\mathcal{V}$, *we get* $\mathcal{Z}_\mu \subseteq ker\pi_\mu$. *Now let* $w \in \mathcal{W}_\mu$ *and* $\pi_\mu(w) = 1$. *By (3.24) and Proposition 3.30, we can write* $w = \dot{w}z(\prod_{i\in J_s} t^{(i)}_{k\alpha_i})(\prod_{j\in J_\ell} t^{(j)}_{\beta_j})$ *where* $\dot{w} \in \dot{\mathcal{W}}$, α_i*'s*$\in \dot{Q}_s$, β_j*'s*$\in \dot{Q}_\ell$ *and* $z \in \mathcal{Z}_\mu$. *So*

$$1 = \pi_\mu(w) = \dot{w}(\prod_{i\in J_s} t^{(i)}_{k\alpha_i})(\prod_{j\in J_\ell} t^{(j)}_{\beta_j})_{|\mathcal{V}}.$$

By Proposition 3.30, $\dot{w} = 1 = (\prod_{i\in J_s} t^{(i)}_{k\alpha_i})(\prod_{j\in J_\ell} t^{(j)}_{\beta_j})$. *Then using an argument similar to that in the proof of Proposition 3.30, to show* $\tilde{H}$ *is torsion free, we get* $\alpha_i = 0 = \beta_j$ *for all* $i \in J_s$ *and* $j \in J_\ell$. *Thus* $w = z \in \mathcal{Z}_\mu$.

4 The Notion of a Basis

The notion of a basis for a finite or affine root system or in general for the root system of a Kac-Moody algebra plays a crucial role in the related theory. A basis Π for such root systems is a finite subset of roots with respect to which the set of roots can be decomposed into the disjoint union of, so called, positve and negative roots. A positive root is a linear combination of the roots in Π with all coeffficeints in $\mathbb{Z}_{\geq 0}$. A negative root is the negative of a positive root. The reflections on roots in Π generate the whole Weyl group of the corresponding root system and the Weyl group acts transitively on the set of bases. All bases have the same cardinality.

For an extended affine root system R with nullity $\nu \geq 2$. having a notion of a basis with characteristics similar to those of a finite or affine root system is far from being realized. Indeed for such root systems there is in general no finite subset Π of R so that any root is a $\mathbb{Z}-$linear combination of roots in Π with all coefficients having the same sign. This being said, we are not planning to define a notion of basis for an EARS, instead we will show that one of the important characteristics of finite and affine root systems holds for the class of EARS's. Indeed, for a reduced EARS R we would like to consider the problem of existence of a finite subset Π of $R^\times$ so that the action of the subgroup of $GL(\tilde{\mathcal{V}})$ generated by reflections based on roots in Π gives all nonisotropic roots, Π having least cardinality with this property.

First we need to introduce a general setup which we use throughout. Recall that $\dot{R}^\times = \dot{R}_{sh} \cup \dot{R}_{lg}$ and $\dot{\Pi} = \{\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell\}$ is a set of simple roots for $\dot{R}$. For a subset P of $\mathcal{V}$, consisting of nonisotropic elements we set

$$
\begin{aligned}
&\mathcal{W}_P = \langle r_\alpha : \alpha \in P\rangle \leq GL(\tilde{\mathcal{V}}),\\
&R_P^\times = \mathcal{W}_P P,\\
&S_P = \{\delta \in \mathcal{V}^0 : \alpha + \delta \in R_P^\times \text{ for some } \alpha \in \dot{R}_{sh}\},\\
&L_P = \{\delta \in \mathcal{V}^0 : \alpha + \delta \in R_P^\times \text{ for some } \alpha \in \dot{R}_{lg}\} \quad \textit{and}\\
&R_P = (S_P + S_P) \cup R_P^\times.
\end{aligned} \tag{4.1}
$$

In what follows we fix

$$
P = \{\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell, \beta_1 + \sigma_1, \ldots, \beta_m + \sigma_m,\ \beta_i \in \dot{R}^\times, \sigma_i \in \langle S\rangle, \sigma_i\text{'s spans } \mathcal{V}^0\}. \tag{4.2}
$$

Note that $\dot{\Pi} \subseteq P \subseteq \bar{R}^{\times}$. Thus

$$\dot{\mathcal{W}} \subseteq \mathcal{W}_P \subseteq \bar{\mathcal{W}} \quad \text{and} \quad R_P^{\times} \subseteq \bar{R}^{\times}. \tag{4.3}$$

Lemma 4.4 $R_P^{\times} = (\dot{R}_{sh} + S_P) \cup (\dot{R}_{lg} + L_P)$.

Proof. First we show $R_P^{\times} \subseteq (\dot{R}_{sh}+S_P)\cup(\dot{R}_{lg}+L_P)$. Let $\beta \in P$ and $w = r_{\gamma_1} \cdots r_{\gamma_n} \in \mathcal{W}_P$, γ_i's$\in P$. We must show $w(\beta) \in (\dot{R}_{sh} + S_P) \cup (\dot{R}_{lg} + L_P)$. We can write $\beta = \dot{\beta} + \delta$ and $\gamma_i = \dot{\gamma}_i + \lambda_i$ where $\dot{\beta}, \dot{\gamma}_i \in \dot{R}^{\times}$ and $\delta, \lambda_i \in \mathcal{V}^0$. Then it is easy to see that

$$w(\beta) = r_{\dot{\gamma}_1} \cdots r_{\dot{\gamma}_n}(\dot{\beta}) + \sigma \quad \text{for some } \sigma \in \mathcal{V}^0.$$

By the definition of $R_P^{\times}$, we have $w(\beta) \in R_P^{\times}$. But then from the definition of S_P and L_P we have $\sigma \in S_P$ or $\sigma \in L_P$. Thus $w(\beta) \in (\dot{R}_{sh} + S_P) \cup (\dot{R}_{lg} + L_P)$. This gives the desired inclusion. We now show the reverse inclusion by proving that $\dot{R}_{sh} + S_P \subseteq R_P^{\times}$ and $\dot{R}_{lg} + L_P \subseteq R_P^{\times}$. Let $\delta \in S_P$. By the definition of S_P, there exists $\alpha \in \dot{R}_{sh}$ such that $\alpha + \delta \in R_P^{\times}$. Since $\dot{\mathcal{W}} \subseteq \mathcal{W}_P$ we have

$$\dot{R}_{sh} + \delta = \dot{\mathcal{W}}\alpha + \delta = \dot{\mathcal{W}}(\alpha + \delta) \subseteq \dot{\mathcal{W}}R_P^{\times} = \dot{\mathcal{W}}\mathcal{W}_P P = \mathcal{W}_P P = R_P^{\times}.$$

Since $\delta \in S_P$ was arbitrary we get $\dot{R}_{sh} + S_P \subseteq R_P^{\times}$. If $\lambda \in L_P$, then there exists $\beta \in \dot{R}_{lg}$ so that $\beta + \lambda \in R_P^{\times}$. Thus

$$\dot{R}_{lg} + \lambda = \dot{\mathcal{W}}\beta + \lambda = \dot{\mathcal{W}}(\beta + \lambda) \subseteq \dot{\mathcal{W}}R_P^{\times} = \dot{\mathcal{W}}\mathcal{W}_P P = \mathcal{W}_P P = R_P^{\times}.$$

Thus $\dot{R}_{lg} + L_P \subseteq R_P^{\times}$. This completes the proof of the lemma. □

Lemma 4.5 $\mathcal{W}_P = \mathcal{W}_{R_P^{\times}}$.

Proof. We have $\mathcal{W}_P \subseteq \mathcal{W}_{R_P^{\times}}$. Now to see the reverse inclusion, let $\alpha \in R_P^{\times} = \mathcal{W}_P P$. So there exists $w \in \mathcal{W}_P$ and $\beta \in P$ so that $\alpha = w\beta$. Then

$$r_\alpha = r_{w\beta} = w r_\beta w^{-1} \in \mathcal{W}_P.$$

Thus $\mathcal{W}_P$ contains all generators of $\mathcal{W}_{R_P^{\times}}$. Hence $\mathcal{W}_{R_P^{\times}} \subseteq \mathcal{W}_P$. □

Lemma 4.6 *If $\dot{R}_{lg} \neq \emptyset$, then*

$$kS_P + L_P \subseteq L_P \quad \textit{and} \quad S_P + L_P \subseteq S_P.$$

Proof. Let $\delta \in S_P$ and $\lambda \in L_P$. There exist roots $\alpha \in \dot{R}_{sh}$ and $\beta \in \dot{R}_{lg}$ such that $(\alpha, \check{\beta}) = -1, (\beta, \check{\alpha}) = -k$. By Lemma 4.4, $\alpha+\delta, \beta+\lambda \in R_P^\times$ and so $r_{\alpha+\delta}, r_{\beta+\lambda} \in \mathcal{W}_{R_P^\times} = \mathcal{W}_P$. Thus

$$\alpha + \beta + \lambda + \delta = r_\beta(\alpha) + \lambda + \delta = r_{\beta+\lambda}(\alpha + \delta) \in \mathcal{W}_P R_P^\times \subseteq R_P^\times$$

and

$$\beta + k\alpha + k\delta + \lambda = r_\alpha(\beta) + \lambda + k\delta = r_{\alpha+\delta}(\beta + \lambda) \in \mathcal{W}_P R_P^\times \subseteq R_P^\times.$$

Since $r_\beta(\alpha) \in \dot{R}_{sh}$ and $r_\alpha(\beta) \in \dot{R}_{lg}$, Lemma 4.4 gives $\lambda + \delta \in S_P$ and $k\delta + \lambda \in L_P$. It follows that

$$S_P + L_P \subseteq S_P \quad \text{and} \quad kS_P + L_P \subseteq L_P.$$

□

Lemma 4.7 *(i) S_P and L_P are semilattices.*

(ii) If $\dot{R}$ is not of type A_1 or B_ℓ, then S_P is a lattice.

(iii) If $\dot{R}$ is not of type $C_\ell (l \geq 3)$ or B_2, then L_P is a lattice.

Proof. (i) We must show S_P and L_P satisfy (S1)-(S4) (see Definition I.1.23).

(S1) is clear. For (S2) let $\delta \in S_P$ and $\alpha \in \dot{R}_{sh}$. Then $-\alpha - \delta = r_{\alpha+\delta}(\alpha + \delta) \in \mathcal{W}_{R_P^\times} R_P^\times \subseteq R_P^\times$, so $-\delta \in S_P$. Therefore $S_P = -S_P$. Now let $\lambda, \delta \in S_P$ and $\alpha \in \dot{R}_{sh}$. Then

$$-\alpha + \lambda = r_{\alpha+\delta}(\alpha + \lambda) - 2\delta \in R_P^\times.$$

From this it follows that $S_P \pm 2S_P \subseteq S_P$. The inclusion $L_P \pm 2L_P \subseteq L_P$ follows using an analogous argument, so (S2) holds. We now consider (S3). From (4.2) we see that for any i, $1 \leq i \leq m$ we have either $\sigma_i \in S_P$ or $\sigma_i \in L_P$. Since $L_P \subseteq S_P$ we get $\sigma_i \in S_P$ for $1 \leq i \leq m$. So $2\sigma_i \in 2S_P \subseteq L_P$ for $1 \leq i \leq m$. Since σ_i's spans $\mathcal{V}^0$, both S_P and L_P span $\mathcal{V}^0$. For (S4) note that by (4.3) and Lemma 4.4,

$$(\dot{R}_{sh} + S_P) \cup (\dot{R}_{lg} + L_P) = R_P^\times \subseteq \tilde{R}^\times = (\dot{R}_{sh} + \langle S \rangle) \cup (\dot{R}_{lg} + \langle L \rangle).$$

Thus $S_P \subseteq \langle S \rangle$ and $L_P \subseteq \langle L \rangle$. So, as subsets of discrete sets both $S_P + S_P$ and $L_P + L_P$ are discrete.

(ii) In this case there are roots $\alpha, \beta \in \dot{R}_{sh}$ such that $(\alpha, \check{\beta}) = -1$. Therefore if $\delta, \lambda \in S_P$, then $\alpha + \delta, \beta + \lambda \in R_P^\times$ and so

$$r_{\alpha+\delta}(\beta + \lambda) = \alpha + \beta + (\delta + \lambda) = r_\alpha(\beta) + (\delta + \lambda) \in (\dot{R}_{sh} + \mathcal{V}^0) \cap R_P^\times.$$

By Lemma 4.4, $\delta + \lambda \in S_P$. Thus $S_P + S_P \subseteq S_P$ and so S_P is a lattice.

(iii) In this case there exist roots $\alpha, \beta \in \dot{R}_{lg}$ such that $(\alpha, \check{\beta}) = -1$. Now using a similar argument as in part (ii) we see that L_P is a lattice. □

By Lemmas 4.4, 4.7 and Theorem I.1.28 we have

Proposition 4.8 *R_P is an EARS of the same type of R and of nullity ν. Moreover*

$$R_P = (S_P + S_P) \cup (\dot{R}_{sh} + S_P) \cup (\dot{R}_{lg} + L_P).$$

□

Definition 4.9 *Let $P \subseteq R^\times$. We say a nontrivial element $w \in \mathcal{W}_P$ has length m (with respect to P), written $l(w) = m$, if m is the smallest integer so that w can be written in the form*

$$w = r_{\alpha_1} \cdots r_{\alpha_m}, \quad \alpha_i \in P \text{ for } 1 \le i \le m. \tag{4.10}$$

We define $l(1) = 0$.

Let θ_s and θ_ℓ be the highest short and the highest long roots of $\dot{R}$ with respect to $\dot{\Pi}$, respectively (for simply laced cases $\theta_s = \theta$). By Convention 2.1, for simply laced cases we have only the highest short root $\theta_s (:= \theta)$. We set

$$\begin{gathered} \alpha_i = \dot{\alpha}_i \quad \text{for } 1 \le i \le \ell, \\ \alpha_{\ell+i} = \delta_i - \theta_s \text{ for } i \in J_s, \qquad \alpha_{\ell+i} = \delta_i - \theta_\ell \text{ for } i \in J_\ell \quad \text{and} \\ \Pi := \{\alpha_1, \ldots, \alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_{\ell+\nu}\}. \end{gathered} \tag{4.11}$$

We note that for simply laced cases $\alpha_{\ell+i} = \delta_i - \theta$ for all $i \in J$. We also note that Π is of the form (4.2). Therefore by Proposition (4.8), R_Π is an EARS and

$$R_\Pi = (S_\Pi + S_\Pi) \cup (\dot{R}_{sh} + S_\Pi) \cup (\dot{R}_{lg} + L_\Pi).$$

Clearly $\Pi \subseteq R^\times$ and $\mathcal{W}_\Pi \subseteq \mathcal{W}$, so $R^\times_\Pi = \mathcal{W}_\Pi\Pi \subseteq \mathcal{W}R^\times \subseteq R^\times$. By Lemma 4.4, $S_\Pi \subseteq S$ and $L_\Pi \subseteq L$. Therefore $R_\Pi \subseteq R$. Moreover from Lemma 4.6 we get $\delta_1,\dots,\delta_\nu \in S_\Pi$ and $k\delta_1,\dots,k\delta_t,\delta_{t+1},\dots,\delta_\nu \in L_\Pi$. Thus

$$\langle S_\Pi\rangle = \langle S\rangle \quad \text{and} \quad \langle L_\Pi\rangle = \langle L\rangle. \tag{4.12}$$

Proposition 4.13 *R_Π is an extended affine root system of the same type and the same twist number as R. Moreover $R = R_\Pi$ if R is not of type $X = A_1,\ B_\ell,\ C_\ell$.*

Proof. The first statement follows immediately from Proposition 4.8 and (4.12). If $X \neq A_1, B_\ell, C_\ell$, then by Theorem I.1.28, S, S_Π, L and L_Π are lattices. Therefore Proposition 4.8 and (4.12) gives $R_\Pi = R$. □

From Lemma 4.5 and Proposition 4.13 we have

Corollary 4.14 *If $X \neq A_1, B_\ell, C_\ell$, then $R_\Pi = R = \tilde{R}$ and $\mathcal{W}_\Pi = \mathcal{W} = \tilde{\mathcal{W}}$.*

In section 1, we defined the notion of index for a semilattice (see Definition 1.19). We now want to define a notion of index for an EARS R. This notion as we shall see depends on the rank, nullity, twist number and the index of semilattices which arise in the structure of R. Recall from (2.3), Lemma 1.26 and (3.1) that if R has type B_ℓ or C_ℓ, then

$$R = (S+S)\cup(\dot{R}_{sh}+S)\cup(\dot{R}_{lg}+L).$$

$$\begin{gathered}\text{there exists subspaces } \mathcal{V}^0_1 \text{ and } \mathcal{V}^0_2 \text{ of } \mathcal{V}^0 \text{ and}\\ \textit{semilattices } S_1 \text{ in } \mathcal{V}^0_1 \text{ and } S_2 \text{ in } \mathcal{V}^0_2 \text{ so that}\\ \mathcal{V}^0 = \mathcal{V}^0_1 \oplus \mathcal{V}^0_2, \quad dim\mathcal{V}^0_1 = t, \quad dim\mathcal{V}^0_2 = \nu - t,\\ S = S_1 \oplus \langle S_2\rangle, \quad L = 2\langle S_1\rangle \oplus S_2,\\ \langle S_1\rangle = \textstyle\sum_{i=1}^t \mathbf{Z}\delta_i \quad \text{and} \quad \langle S_2\rangle = \sum_{i=t+1}^\nu \mathbf{Z}\delta_i \text{ where}\\ \delta_1,\dots,\delta_t \in S_1 \quad \text{and} \quad \delta_{t+1},\dots,\delta_\nu \in S_2.\end{gathered} \tag{4.15}$$

By Remark 1.14, we can write

$$\begin{aligned}&S_1 = \textstyle\bigcup_{i=0}^{ind(S_1)}(\tau_i + 2\langle S_1\rangle) \quad \text{where}\\ &\tau_i\text{'s represent distinct cosets of } 2\langle S_1\rangle \text{ in } \langle S_1\rangle \quad \textit{and}\\ &\tau_0 = 0, \tau_1 = \delta_1,\dots,\tau_t = \delta_t.\end{aligned} \tag{4.16}$$

Also

$$S_2 = \bigcup_{i=0}^{ind(S_2)} (\sigma_i + 2\langle S_2 \rangle) \quad \text{where}$$
$$\sigma_i\text{'s represent distinct cosets of } 2\langle S_2\rangle \text{ in } \langle S_2\rangle \quad \textit{and} \tag{4.17}$$
$$\sigma_0 = 0, \sigma_1 = \delta_{t+1}, \ldots, \sigma_{\nu-t} = \delta_\nu.$$

So

$$ind(S_1) \geq t \quad \text{and} \quad ind(S_2) \geq \nu - t. \tag{4.18}$$

Note that if $1 \leq i \leq ind(S_1)$ and $\tau_i \in \langle L \rangle$, then $\tau_i \in \langle L \rangle \cap S_1 = (2\langle S_1 \rangle \oplus \langle S_2 \rangle) \cap S_1 \subseteq 2\langle S_1 \rangle$ which contradicts the choice of τ_i's in (4.16). Also if $1 \leq i \leq ind(S_2)$ and $\sigma_i \in 2\langle S \rangle$, then $\sigma_i \in 2\langle S \rangle \cap S_2 = 2(\langle S_1 \rangle \oplus \langle S_2 \rangle) \cap S_2 \subseteq 2\langle S_2 \rangle$ which contradicts the choice of σ_i's in (4.17). Thus

$$\tau_i \notin \langle L \rangle \quad \text{for } 1 \leq i \leq ind(S_1) \quad \text{and} \quad \sigma_i \notin 2\langle S \rangle \quad \text{for } 1 \leq i \leq ind(S_2). \tag{4.19}$$

Let $\dot{R}'$ be another choice of the finite root system in §1 and $R = (S' + S') \cup (\dot{R}'_{sh} + S') \cup (\dot{R}'_{lg} + L')$ be the corresponding expression of R in the form (1.8). Then using Lemma 1.26 we have

$$S' = S'_1 \oplus \langle S'_2 \rangle \quad \text{and} \quad L' = 2\langle S' \rangle \oplus S'_2$$

where S'_1 and S'_2 are semilattices in some subspaces of $\mathcal{V}^0$ of dimensions t and $\nu - t$, respectively.

Lemma 4.20 *$ind(S_1) = ind(S'_1)$ and $ind(S_2) = ind(S'_2)$.*

Proof. By Lemmas 1.9 and 1.20, we have $ind(S) = ind(S')$ and $ind(L) = ind(L')$. Therefore, by Lemma 1.21,

$$ind(S_1)ind(\langle S_2 \rangle) + ind(S_1) + ind(\langle S_2 \rangle) = ind(S'_1)ind(\langle S'_2 \rangle) + ind(S'_1) + ind(\langle S'_2 \rangle).$$

Since both lattices $\langle S_2 \rangle$ and $\langle S'_2 \rangle$ have rank t, $ind(\langle S_2 \rangle) = ind(\langle S'_2 \rangle)$. Thus $ind(S_1) = ind(S'_1)$. Using $ind(L) = ind(L')$ and a similar argument as above we get $ind(S_2) = ind(S'_2)$. □

Definition 4.21 *Let R be an EARS of type X. If X is simply laced of rank > 1 or is of type F_4 or G_2, we define index of R, written $ind(R)$, to be zero. If $X = A_1$ and*

$R = (S+S)\cup(\dot{R}+S)$ is an expression of R in the form (1.7), we define $ind(R) := ind(S)-\nu$. By Lemmas 1.9 and 1.20, $ind(R)$ is a well-defined integer and by Remark 1.17, $ind(R)\geq 0$. If $X = B_\ell, C_\ell$, and R has an expression of the form (4.15) we define $ind(R) := ind(S_1)+ind(S_2)-\nu$ if $X = B_2$, $ind(R) := ind(S_1)-t$ if $B = B_\ell(l\geq 3)$ and $ind(R) := ind(S_2)-(\nu-t)$ if $X = C_\ell(l\geq 3)$. By Lemma (4.20), $ind(R)$ is well-defined and by (4.18), $ind(R)\geq 0$.

Let $\check{R}$ be the dual root system of R defined by (2.8).

Lemma 4.22 *$ind(R) = ind(\check{R})$.*

Proof. If R is simply laced, we have $R=\check{R}$ and so $ind(R)=ind(\check{R})$. Since $\check{R}$ has the same type of R for the cases F_4 and G_2, so $ind(R)=0=ind(\check{R})$. Therefore, we can assume that R is of type B_ℓ or C_ℓ. Then, we have $S = S_1 \dotplus \langle S_2\rangle$ and $L = 2\langle S_1\rangle \dotplus S_2$, where S_1 and S_2 are semilattices in $\mathcal{V}_1^0$ and $\mathcal{V}_2^0$ as in (4.15). By Proposition 2.9, we have

$$\check{R} = (\frac{1}{2}L+\frac{1}{2}L)\cup(\frac{1}{2}\dot{R}_{lg}+\frac{1}{2}L)\cup(\dot{R}_{sh}+S).$$

Let $S_1' := (1/2)S_2$ and $S_2' := S_1$. Then S_1' and S_2' are semilattices in $\mathcal{V}_2^0$ and $\mathcal{V}_1^0$, respectively. Moreover

$$\frac{1}{2}L = S_1' \dotplus \langle S_2'\rangle \quad\text{and}\quad S = 2\langle S_1'\rangle \dotplus S_2'.$$

If R has type B_2, then by Definition 4.21, $ind(R) = ind(S_1)+ind(S_2)-\nu = ind(S_2')+ind(2S_1')-\nu = ind(S_2')+ind(S_1')-\nu = ind(\check{R})$. If R has type $B_\ell(l\geq 3)$, then $\check{R}$ has type C_ℓ and twist number $\nu-t$. So $ind(R)=ind(S_1)-t = ind(S_2')-t = ind(S_2')-(\nu-(\nu-t)) = ind(\check{R})$. Finally if R has type $C_\ell(l\geq 3)$, then $\check{R}$ has type B_ℓ with twist number $\nu-t$. So $ind(R)=ind(S_2)-(\nu-t)=ind(2S_1')-(\nu-t)=ind(S_1')-(\nu-t)=ind(\check{R})$. □

We are now ready to state the main result of this section. We denote by $|P|$, the cardinality of a set P. Recall from section 1 the projection map $\bar{}:\mathcal{V}\to\bar{\mathcal{V}}$ and recall that $\bar{R}$ is a finite root system in $\bar{\mathcal{V}}$.

Theorem 4.23 *Let R be a reduced EARS of type X. Then there exists a subset $\Pi(X)$ of $R^\times$ of cardinality $ind(R)+l+\nu$ so that $\mathcal{W}_{\Pi(X)}\Pi(X) = R^\times$. Moreover if Π' is any subset of $R^\times$ so that*

$$\bar{\Pi}' \text{ contains a basis of } \bar{R} \tag{4.24}$$

and $\mathcal{W}_{\Pi'}\Pi' = R^{\times}$, *then* $|\Pi'| \geq ind(R) + l + \nu$.

As we shall see in the sequel we use the assumption (4.24) only for the cases $B_\ell(l \geq 3)$ and $C_\ell(l \geq 3)$ and the other cases will be proved without using (4.24). We also note that the assumption (4.24) does not impose any restriction on the sign of roots in Π', since $\mathcal{W}_{\Pi'}\Pi' = \mathcal{W}_P P$ where P is any set obtained from Π' by changing the sign of some roots in Π'. To proceed with the proof, we consider each of the cases (a) X is simply laced with rank > 1, $X = F_4, G_2$, (b) $X = A_1$, (c) $X = B_2$, (d) $X = B_\ell(l \geq 3)$ and (e) $X = C_\ell(l \geq 3)$ separately and in each case we state some lemmas and propositions to complete the proof of theorem.

(a) $\mathbf{X}$ = simply laced of rank > 1 or $\mathbf{X = F_4, G_2}$.

Let Π be the set defined by (4.11). Then $|\Pi| = l + \nu$.

Lemma 4.25 $\mathcal{W}_\Pi\Pi = R^{\times}$. *Moreover if* $\Pi' \subseteq R^{\times}$ *and* $\mathcal{W}_{\Pi'}\Pi' = R^{\times}$, *then* $|\Pi'| \geq l + \nu$.

Proof. We have already seen the first statement in Proposition 4.13. Now let Π' is as in the statement. Then

$$\mathcal{V} = \mathrm{Span}_{\mathbb{R}} R^{\times} = \mathrm{Span}_{\mathbb{R}} \mathcal{W}_{\Pi'}\Pi' = \mathrm{Span}_{\mathbb{R}}\Pi'.$$

Since $\dim\mathcal{V} = l + \nu$, we get $|\Pi'| \geq l + \nu$. $\square$

(b) $\mathbf{X = A_1}$.

We have $R = (S + S) \cup (\dot{R} + S)$ where $\langle S\rangle = \sum_{i=1}^{\nu} \mathbb{Z}\delta_i$ and $\delta_1, \ldots, \delta_\nu \in S$ (see (3.1)). Thus $ind(S) \geq \nu$ (since $\delta_1, \ldots, \delta_\nu \notin 2\langle S\rangle$). By Remark 1.17, we can write

$$S = \bigcup_{i=0}^{ind(S)}(\tau_i + 2\langle S\rangle) \quad \text{where } \tau_i\text{'s represent distinct cosets of } 2\langle S\rangle \text{ in } \langle S\rangle \quad \text{and} \quad \tau_0 = 0, \tau_1 = \delta_1, \ldots, \tau_\nu = \delta_\nu. \tag{4.26}$$

Let $\dot{R} = \{-\alpha, 0, \alpha\}$. We set

$$\Pi(A_1) = \{\alpha, \tau_1 - \alpha, \ldots, \tau_{ind(S)} - \alpha\}.$$

$\Pi(A_1)$ has cardinality $1 + ind(S) = ind(R) + 1 + \nu$. Recall definitions of $\mathcal{W}_{\Pi(A_1)}$, $S_{\Pi(A_1)}$ and $R_{\Pi(A_1)}$ from (4.1).

Proposition 4.27 $\mathcal{W}_{\Pi(A_1)}\Pi(A_1) = R^\times$. *Moreover if* $\Pi' \subseteq R^\times$ *and* $\mathcal{W}_{\Pi'}\Pi' = R^\times$, *then* $|\Pi'| \geq ind(R) + 1 + \nu$.

Proof. By the way $R^\times_{\Pi(A_1)}$ is defined, we have $\mathcal{W}_{\Pi(A_1)}\Pi(A_1) = R^\times_{\Pi(A_1)}$. Clearly the set $\Pi(A_1)$ satisfies (4.2). By Lemma 4.7 and Proposition 4.8, $S_{\Pi(A_1)}$ is a semilattice in $\mathcal{V}^0$ and $R_{\Pi(A_1)} = (S_{\Pi(A_1)} + S_{\Pi(A_1)}) \cup (\dot{R} + S_{\Pi(A_1)})$. So it is enough to show that $S = S_{\Pi(A_1)}$. We have

$$(\dot{R} \setminus \{0\}) + S_{\Pi(A_1)} = R^\times_{\Pi(A_1)} = \mathcal{W}_{\Pi(A_1)}\Pi(A_1) \subseteq \mathcal{W}R^\times \subseteq R^\times = (\dot{R} \setminus \{0\}) + S,$$

so $S_{\Pi(A_1)} \subseteq S$. Now we show that $S \subseteq S_{\Pi(A_1)}$. Since $\delta_i \in S_{\Pi(A_1)}$ for $1 \leq i \leq \nu$, we have $\langle S_{\Pi(A_1)} \rangle = \langle S \rangle$. Then for each i, $1 \leq i \leq m$, we have

$$\tau_i + 2\langle S \rangle = \tau_i + 2\langle S_{\Pi(A_1)} \rangle \subseteq S_{\Pi(A_1)} + 2\langle S_{\Pi(A_1)} \rangle \subseteq S_{\Pi(A_1)}.$$

Thus $S = \cup_{i=1}^{ind(S)}(\tau_i + 2\langle S \rangle) \subseteq S_{\Pi(A_1)}$. This finishes the proof of the first statement.

For the second statement let $\Pi' \subseteq R^\times$ and $\mathcal{W}_{\Pi'}\Pi' = R^\times$. Then $\Pi' = \{\dot{\beta}_i + \eta_i\}_{i \in I}$ for some index set I, where $\dot{\beta}_i = \pm\alpha$ and $\eta_i \in S$ for $i \in I$. We claim that $S = \cup_{i \in I}(\eta_i + 2\langle S \rangle)$. Since $\eta_i \in S$, we have $\eta_i + 2\langle S \rangle \subseteq S + 2\langle S \rangle \subseteq S$, so $\cup_{i \in I}(\eta_i + 2\langle S \rangle) \subseteq S$. On the other hand let $\delta \in S$. So $\alpha + \delta \in R^\times = \mathcal{W}_{\Pi'}\Pi'$. Therefore $\alpha + \delta = w(\dot{\beta}_i + \eta_i)$ for some $w \in \mathcal{W}_{\Pi'}$ and some $i \in I$. By Lemma 2.18(i), we have

$$\alpha + \delta = w(\dot{\beta}_i + \eta_i) \in \dot{R}^\times + \eta_i + 2\langle S \rangle.$$

Thus $\delta \in \eta_i + 2\langle S \rangle$. So $S \subseteq \cup_{i \in I}(\eta_i + 2\langle S \rangle)$. This gives $S = \cup_{i \in I}(\eta_i + 2\langle S \rangle)$. Because of the choice of τ_i's (see (4.26)) we get $|\Pi'| \geq ind(S) + 1 = ind(R) + 1 + \nu$. This completes the proof. □

Now we start the discussion of types $X = B_\ell (l \geq 2)$, $C_\ell (l \geq 3)$. First we establish some results which we apply to both cases $X = B_\ell, C_\ell$. Later we consider each of these cases separately.

Lemma 4.28 *Let* $\Pi' \subseteq R^\times$ *and* $\mathcal{W}\Pi' = R^\times$.

(i) If $X = B_\ell (l \geq 2)$, *then* Π' *contains a subset* Π'_1 *so that* $|\Pi'_1| = ind(S_1)$ *and elements of* Π'_1 *are of the form* $\beta + \eta$ *where* $\beta \in \dot{R}_{sh}$ *and* $\eta \in \tau_i + \langle L \rangle$ *for some* i, $1 \leq i \leq ind(S_1)$.

(ii) If $X = B_2$ or $X = C_\ell (l \geq 3)$, then Π' contains a subset Π'_2 so that $|\Pi'_2| = ind(S_2)$ and elements of Π'_2 are of the form $\dot\beta + \eta$ where $\dot\beta \in \dot R_{lg}$ and $\eta \in \sigma_i + 2\langle S\rangle$ for some i, $1 \leq i \leq ind(S_2)$.

Proof. Let $\Pi' = \{\dot\beta_i + \eta_i\}_{i\in I}$, for some index set I, where $\dot\beta_i$'s$\in \dot R^\times$ and η_i's$\in S$. We first prove (i). Let $1 \leq i \leq ind(S_1)$. Then $\tau_i - \theta_s \in R^\times = \mathcal{W}\Pi'$. So there exist $w \in \mathcal{W}$ and $k_i \in I$ so that $\tau_i - \theta_s = w(\dot\beta_{k_i} + \eta_{k_i})$. Clearly $\dot\beta_{k_i} \in \dot R_{sh}$. By Lemma 2.18(ii), we have

$$\dot\beta_{k_i} + \eta_{k_i} = w^{-1}(\tau_i - \theta_s) = \tau_i - w^{-1}(\theta_s) \in \dot R_{sh} + \tau_i + \langle L\rangle.$$

Thus $\eta_{k_i} \in \tau_i + \langle L\rangle$. We claim that if $1 \leq i, j \leq ind(S_1)$ and $i \neq j$, then $k_i \neq k_j$. Because, if otherwise $k_i = k_j$ then $\eta_{k_i} = \eta_{k_j}$ and so $\tau_i - \tau_j \in \langle L\rangle = 2\langle S_1\rangle \oplus \langle S_2\rangle$. This and the fact that $\tau_i - \tau_j \in S_1 - S_1 \subseteq \langle S_1\rangle$ give $\tau_i - \tau_j \in 2\langle S_1\rangle$. But this contradicts the choice of τ_i's in (4.16). Hence if $i \neq j$, $1 \leq i, j \leq ind(S_1)$, then $k_i \neq k_j$. Therefore the set $\Pi'_1 = \{\dot\beta_{k_i} + \eta_{k_i} : 1 \leq i \leq ind(S_1)\}$ satisfies the statement. This finishes the proof of part (i).

(ii) Let $1 \leq i \leq ind(S_2)$. Then $\sigma_i - \theta_\ell \in R^\times = \mathcal{W}\Pi'$. So there exists $w \in \mathcal{W}$ and $k_i \in I$ so that $\sigma_i - \theta_\ell = w(\dot\beta_{k_i} + \eta_{k_i})$. Clearly $\dot\beta_{k_i} \in \dot R_{lg}$. By Lemma 2.18(iii), we have

$$\dot\beta_{k_i} + \eta_{k_i} = w^{-1}(\sigma_i - \theta_\ell) = \sigma_i - w^{-1}(\theta_\ell) \in \dot R_{lg} + \sigma_i + 2\langle S\rangle.$$

So $\eta_{k_i} \in \sigma_i + 2\langle S\rangle$. We claim that if $1 \leq i, j \leq ind(S_2)$ and $i \neq j$ then, $k_i \neq k_j$. Because, otherwise if $k_i = k_j$, then $\eta_{k_i} = \eta_{k_j}$ and so $\sigma_i - \sigma_j \in 2\langle S\rangle = 2\langle S_1\rangle \oplus 2\langle S_2\rangle$. But this and the fact that $\sigma_i - \sigma_j \in S_2 - S_2 \subseteq \langle S_2\rangle$ implies that $\sigma_i - \sigma_j \in 2\langle S_2\rangle$ which contrsdicts the choice of σ_i's in (4.17). Therefore the set $\Pi'_2 = \{\dot\beta_{k_i} + \eta_{k_i} : 1 \leq i \leq ind(S_2)\}$ satisfies the statement. This completes the proof of part (ii). □

Corollary 4.29 *Let Π, Π'_1 and Π'_2 are as in Lemma 4.28.*

(i) If $X = B_\ell (l \geq 2)$, then $\Pi' \setminus \Pi'_1$ contains at least one short root.

(ii) If $X = B_2$ or $X = C_\ell (l \geq 3)$, then $\Pi' \setminus \Pi'_2$ contains at least one long root.

Proof. (i) Suppose to the contrary that the only short roots of Π' are elements of Π'_1. Since $\theta_s \in R^\times = \mathcal{W}\Pi'$, there exist $w \in \mathcal{W}$ and $\beta \in \Pi'$ so that $w\beta = \theta_s$. Clearly $\beta \in R_{sh}$. By assumption, $\beta \in \Pi'_1$. According to Lemma 4.28(i), $\beta = \dot\beta + \eta$, where $\dot\beta \in \dot R_{sh}$ and

$\eta \in \tau_i + \langle L \rangle$ for some $1 \leq i \leq ind(S_1)$. Then by Lemma 2.18(ii), we have

$$\theta_s = w(\beta) = w(\dot{\beta} + \eta) \in \dot{R}_{sh} + \eta + \langle L \rangle \subseteq \dot{R}_{sh} + \tau_i + \langle L \rangle.$$

Thus $\tau_i \in \langle L \rangle$ which is a contradiction (see (4.19)). Thus $\Pi' \setminus \Pi'_1$ contains at least one short root.

(ii) Suppose the contrary, that is the only long roots of Π' are elements of Π'_2. Since $\theta_\ell \in R^\times = \mathcal{W}\Pi'$, there exist $w \in \mathcal{W}$ and $\beta \in \Pi'$ so that $w\beta = \theta_\ell$. Clearly $\beta \in R_{lg}$. By assumption, $\beta \in \Pi'_2$. According to Lemma 4.28(ii), $\beta = \dot{\beta} + \eta$ where $\dot{\beta} \in \dot{R}_{lg}$ and $\eta \in \sigma_i + 2\langle S \rangle$ for some $1 \leq i \leq ind(S_2)$. Then by Lemma 2.18(iii),

$$\theta_\ell = w(\beta) = w(\dot{\beta} + \eta) \in \dot{R}_{lg} + \eta + 2\langle S \rangle \subseteq \dot{R}_{lg} + \sigma_i + 2\langle S \rangle.$$

So $\sigma_i \in 2\langle S \rangle$ which is a contradiction (see (4.19)). Thus $\Pi' \setminus \Pi'_2$ contains at least one long root. $\square$

From here we divide our argument for the remaining cases of X.

(c) $\mathbf{X = B_2}$

Recall that $\dot{\Pi} = \{\alpha_1, \alpha_2\}$ is a set of simple roots for the finite root system $\dot{R}$. We set

$$\Pi(B_2) = \{\alpha_1, \alpha_2, \tau_1 - \theta_s, \ldots, \tau_{ind(S_1)} - \theta_s, \sigma_1 - \theta_\ell, \ldots, \sigma_{ind(S_2)} - \theta_\ell\}.$$

Then $|\Pi(B_2)| = 2 + ind(S_1) + ind(S_2) = ind(R) + 2 + \nu$. Recall definitions of $\mathcal{W}_{\Pi(B_2)}$, $R_{\Pi(B_2)}$, $S_{\Pi(B_2)}$ and $L_{\Pi(B_2)}$ from (4.1).

Proposition 4.30 *We have $\mathcal{W}_{\Pi(B_2)}\Pi(B_2) = R^\times$. Moreover if $\Pi' \subseteq R^\times$ and $\mathcal{W}_{\Pi'}\Pi' = R^\times$, then $|\Pi'| \geq ind(R) + 2 + \nu$.*

Proof. We have $R^\times_{\Pi(B_2)} = \mathcal{W}_{\Pi(B_2)}\Pi(B_2)$. So we are done if we show that $R^\times = R^\times_{\Pi(B_2)}$. Since $\tau_1 = \delta_1, \ldots, \tau_t = \delta_t$ and $\sigma_1 = \delta_{t+1}, \ldots, \sigma_{\nu-t+1} = \delta_\nu$, $\Pi(B_2)$ satisfies (4.2). So by Lemma 4.8, we have $R^\times_{\Pi(B_2)} = (\dot{R}_{sh} + S_{\Pi(B_2)}) \cup (\dot{R}_{lg} + L_{\Pi(B_2)})$. Thus we only need to show that $S = S_{\Pi(B_2)}$ and $L = L_{\Pi(B_2)}$. We have $\Pi(B_2) \subseteq R^\times$, so $\mathcal{W}_{\Pi(B_2)} \subseteq \mathcal{W}$. Thus $R^\times_{\Pi(B_2)} = \mathcal{W}_{\Pi(B_2)}\Pi(B_2) \subseteq \mathcal{W}R^\times \subseteq R^\times$. So $S_{\Pi(B_2)} \subseteq S$ and $L_{\Pi(B_2)} \subseteq L$. Therefore it only remains to prove $S \subseteq S_{\Pi(B_2)}$ and $L \subseteq L_{\Pi(B_2)}$. Since $\delta_1, \ldots, \delta_t \in S_{\Pi(B_2)}$, $\delta_{t+1}, \ldots, \delta_\nu \in L_{\Pi(B_2)}$ and $2S_{\Pi(B_2)} \subseteq L_{\Pi(B_2)} \subseteq S_{\Pi(B_2)}$ (see Lemma 4.6), we have

$$\sum_{i=1}^{\nu} \mathbb{Z}\delta_i \subseteq \langle S_{\Pi(B_2)} \rangle \subseteq \langle S \rangle = \sum_{i=1}^{\nu} \mathbb{Z}\delta_i \quad \text{and}$$

$$\sum_{i=1}^{t} 2\mathbf{Z}\delta_i \oplus \sum_{i=t+1}^{\nu} \mathbf{Z}\delta_i \subseteq \langle L_{\Pi(B_2)}\rangle \subseteq \langle L\rangle = \sum_{i=1}^{t} 2\mathbf{Z}\delta_i \oplus \sum_{i=t+1}^{\nu} \mathbf{Z}\delta_i.$$

Thus $\langle S_{\Pi(B_2)}\rangle = \langle S\rangle$ and $\langle L_{\Pi(B_2)}\rangle = \langle L\rangle$. By Lemma 4.6 we have $S_{\Pi(B_2)}+\langle L_{\Pi(B_2)}\rangle \subseteq S_{\Pi(B_2)}$ and $L_{\Pi(B_2)} + 2\langle S_{\Pi(B_2)}\rangle \subseteq L_{\Pi(B_2)}$. Therefore

$$\begin{aligned} S = S_1 \oplus \langle S_2\rangle &= \bigcup_{i=0}^{ind(S_1)} (\tau_i + 2\langle S_1\rangle) + \langle S_2\rangle \subseteq \bigcup_{i=0}^{ind(S_1)} (\tau_i + 2\langle S_1\rangle + \langle S_2\rangle) \\ &= \bigcup_{i=0}^{ind(S_1)} (\tau_i + \langle L\rangle) = \bigcup_{i=0}^{ind(S_1)} (\tau_i + \langle L_{\Pi(B_2)}\rangle) \\ &\subseteq S_{\Pi(B_2)} + \langle L_{\Pi(B_2)}\rangle \subseteq S_{\Pi(B_2)}. \end{aligned}$$

Similarly

$$\begin{aligned} L = 2\langle S_1\rangle \oplus S_2 &= 2\langle S_1\rangle \oplus \bigcup_{i=0}^{ind(S_2)} (\sigma_i + 2\langle S_2\rangle) \subseteq \bigcup_{i=0}^{ind(S_2)} (\sigma_i + 2\langle S_1\rangle + 2\langle S_2\rangle) \\ &= \bigcup_{i=0}^{ind(S_2)} (\sigma_i + 2\langle S\rangle) = \bigcup_{i=0}^{ind(S_2)} (\sigma_i + 2\langle S_{\Pi(B_2)}\rangle) \\ &\subseteq L_{\Pi(B_2)} + 2\langle S_{\Pi(B_2)}\rangle \subseteq L_{\Pi(B_2)}. \end{aligned}$$

Thus $S \subseteq S_{\Pi(B_2)}$ and $L \subseteq L_{\Pi(B_2)}$. This completes the proof of the first statement.

For the second statement note that we have $ind(R) = ind(S_1) + ind(S_2) - \nu$. Then the result follows from Lemma 4.28 and corollary 4.29. □

The above proposition concludes our discussion for type B_2.

(d) $\mathbf{X} = \mathbf{B}_\ell (\mathbf{l} \geq \mathbf{3})$

We have $R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L)$. By Theorem I.1.28, in this case L is a lattice. From Definition 1.19 we have $ind(R) = ind(S_1) - t$. We define

$$\Pi(B_\ell) = \{\alpha_1, \ldots, \alpha_\ell, \tau_1 - \theta_s, \ldots, \tau_{ind(S_1)} - \theta_s, \delta_{t+1} - \theta_\ell, \ldots, \delta_\nu - \theta_\ell\}. \tag{4.31}$$

Then $|\Pi(B_\ell)| = ind(R) + \ell + \nu$.

Lemma 4.32 $\mathcal{W}_{\Pi(B_\ell)}\Pi(B_\ell) = R^\times$.

Proof. We follow the same argument as in the proof of Proposition 4.30. Since $\tau_1 = \delta_1, \ldots, \tau_t = \delta_t$, $\Pi(B_\ell)$ satisfies (4.2). So the equality $\mathcal{W}_{\Pi(B_\ell)}\Pi(B_\ell) = R^\times$ is a consequence of inclusions $S \subseteq S_{\Pi(B_\ell)}$ and $L \subseteq L_{\Pi(B_\ell)}$. Using exactly the same argument as in Proposition

4.30 we get $S \subseteq S_{\Pi(B_\ell)}$ and $\langle L \rangle = \langle L_{\Pi(B_\ell)} \rangle$. But both L and $L_{\Pi(B_\ell)}$ are lattices (see Lemma 4.7), so $L = L_{\Pi(B_\ell)}$. □

Lemma 4.33 *Let Π' be a subset of $R^\times$ which satisfies (4.24) and $\mathcal{W}_{\Pi'}\Pi' = R^\times$. Then Π' contains at least $l - 1 + (\nu - t)$ long roots.*

Proof. By assumption, $\bar{\Pi}'$ contains a basis $\{\bar{\beta}_1, \ldots, \bar{\beta}_\ell\}$ of $\bar{R}$. As is well-known we can assume that $\bar{\beta}_1, \ldots, \bar{\beta}_{\ell-1}$ are long and $\bar{\beta}_\ell$ is short. We consider a fixed preimage $\dot{\beta}_i$ of $\bar{\beta}_i$ in Π'. Let $\dot{\mathcal{V}}' = \text{Span}_{\mathbb{R}}\{\dot{\beta}_1, \ldots, \dot{\beta}_\ell\}$. Then $\mathcal{V} = \dot{\mathcal{V}}' \oplus \mathcal{V}^0$. Let

$$\dot{R}' = \{\dot{\beta} \in \dot{\mathcal{V}}' \mid \dot{\beta} + \delta \in R \text{ for some } \delta \in \mathcal{V}^0\}.$$

As in Section 1 we have $\dot{R}'$ is a finite root system in $\dot{\mathcal{V}}'$ isomorphic to $\bar{R}$ and $\dot{R}' \subseteq R$. Write $\dot{R}' = \dot{R}'_{sh} \cup \dot{R}'_{lg}$, where as usual $\dot{R}'_{sh}$ is the set of short roots and $\dot{R}'_{lg}$ is the set of long roots of $\dot{R}'$. Set

$$S' = \{\delta \in \mathcal{V}^0 \mid \dot{\beta} + \delta \in R, \text{ for some } \dot{\beta} \in \dot{R}'_{sh}\} \quad \text{and}$$
$$L' = \{\delta \in \mathcal{V}^0 \mid \dot{\beta} + \delta \in R, \text{ for some } \dot{\beta} \in \dot{R}'_{lg}\}.$$

Then (as in Section 1) we have S' is a semilattice and L' is a lattice in $\mathcal{V}^0$. Moreover

$$R = (S' + S') \cup (\dot{R}'_{sh} + S') \cup (\dot{R}'_{lg} + L').$$

By Lemma 1.9, we have

$$\langle S \rangle = \langle S' \rangle \quad \text{and} \quad L = L'. \tag{4.34}$$

We can write $\Pi' = \{\dot{\beta}_1 + \eta_1, \ldots, \dot{\beta}_\ell + \eta_\ell, \dot{\beta}_{\ell+1} + \eta_{\ell+1}, \ldots\}$ where $\eta_1 = \cdots = \eta_\ell = 0$, $\dot{\beta}_i$'s$\in \dot{R}' \setminus \{0\}$ and η_i's$\in S'$. Since $\mathcal{V} = \text{Span}_{\mathbb{R}} R^\times = \text{Span}_{\mathbb{R}} \mathcal{W}_{\Pi'}\Pi' = \text{Span}_{\mathbb{R}}\Pi'$ and $\dim\mathcal{V} = l + \nu$ so $|\Pi'| \geq l + \nu$. We already know that $\dot{\beta}_1, \ldots, \dot{\beta}_{\ell-1}$ are long roots. If Π' contains another $\nu - t$ long roots we are done. Otherwise there is some integer s, $0 \leq s < \nu - t$ so that Π' contains only $l - 1 + s$ long roots. Without loss of generality assume that the set

$$\Pi'' = \{\dot{\beta}_1 + \eta_1, \ldots, \dot{\beta}_{\ell-1} + \eta_{\ell-1}, \dot{\beta}_{\ell+1} + \eta_{\ell+1}, \ldots, \dot{\beta}_{\ell+s} + \eta_{\ell+s}\}$$

is the set of all long roots of Π'. Let

$$L'' = \text{Span}_{\mathbb{Z}}(\{\eta_i : \ell + 1 \leq i \leq l + s\} \cup \{2\eta_i \mid \dot{\beta}_i + \eta_i \in \Pi' \setminus \Pi''\}).$$

We show $L' \subseteq L''$ by proving that

$$\dot{R}'_{lg} + L' \subseteq \dot{R}'_{lg} + L''. \tag{4.35}$$

Now $R'_{lg} = \dot{R}'_{lg} + L' \subseteq R^\times = \mathcal{W}_{\Pi'}\Pi'$. Since the length is invariant under the action of reflections, we get $R'_{lg} \subseteq \mathcal{W}_{\Pi'}\Pi''$. Therefore to prove (4.35) we only need to show that $\mathcal{W}_{\Pi'}\Pi'' \subseteq \dot{R}'_{lg} + L''$. So let β be fixed in Π''. We use induction on the length of elements in $\mathcal{W}_{\Pi'}$ to prove that

$$\mathcal{W}_{\Pi'}\beta \in \dot{R}'_{lg} + L''. \tag{4.36}$$

We have

$$\beta = \dot{\beta}_i + \eta_i \text{ for some } i \in \{1, \dots, l-1\} \cup \{\ell+1, \dots, l+s\}.$$

Let $w \in \mathcal{W}_{\Pi'}$ have length 1. Then $w = r_\alpha$ for some $\alpha \in \Pi'$. We have $\alpha = \dot{\beta}_j + \eta_j$ for some j. Then

$$r_\alpha(\beta) = r_{\dot{\beta}_j}(\dot{\beta}_i) + (\eta_i + (\beta, \check{\alpha})\eta_j) \in \dot{R}'_{lg} + (\eta_i + (\beta, \check{\alpha})\eta_j).$$

Since $(\beta, \check{\alpha}) \in 2\mathbb{Z}$ for $\alpha \in R_{sh}$ and $(\beta, \check{\alpha}) \in \mathbb{Z}$ for $\alpha \in R_{lg}$, we get $r_\alpha(\beta) \in \dot{R}'_{lg} + L''$. So (4.36) holds. As induction hypothesis assume (4.36) holds for any elements of $\mathcal{W}_{\Pi'}$ with length less than or equal m. Now let $w \in \mathcal{W}_{\Pi'}$ have length m. We are done if we show that for any $\alpha \in \Pi'$, (4.36) holds for the element $r_\alpha w$ of $\mathcal{W}_{\Pi'}$. By induction hypothesis, we have $w(\beta) = \dot{\beta}' + \eta''$ for some $\dot{\beta}' \in \dot{R}'_{lg}$ and $\eta'' \in L''$. Write $\alpha = \dot{\beta}_j + \eta_j$ for some j. Then

$$r_\alpha w(\beta) = r_{\dot{\beta}_j + \eta_j}(\dot{\beta}') + \eta'' = r_{\dot{\beta}_j}(\dot{\beta}') + (\dot{\beta}', \check{\alpha})\eta_j + \eta'' \in \dot{R}'_{lg} + (\dot{\beta}', \check{\alpha})\eta_j + \eta''.$$

Again using the fact that $(\dot{\beta}', \check{\alpha}) \in 2\mathbb{Z}$ if $\alpha \in R_{sh}$ and $(\dot{\beta}', \check{\alpha}) \in \mathbb{Z}$ if $\alpha \in R_{lg}$ we get that $r_\alpha w(\beta) \in \dot{R}'_{lg} + L''$. Therefore (4.36) holds. Thus $L' \subseteq L''$. We claim this is a contradiction. Indeed, let $\Lambda = \sum_{i=1}^{\nu} \mathbb{Z}\delta_i$ and $\bar{\Lambda} = \Lambda/2\Lambda$ be the ν-dimensional $\mathbb{F}_2$–vector space, where here $\mathbb{F}_2$ denotes the field of two elements. Let $\bar{L}'$ and $\bar{L}''$ denote the images of L' and L'' under the projection map $\bar{\ } : \Lambda \to \bar{\Lambda}$, respectively. From definition of L'' and the fact that R has twist number t we have

$$|\bar{L}'| = |\bar{L}| = |\frac{L}{2\Lambda}| = |\frac{L}{2\langle S\rangle}| = 2^{\nu-t} \quad \text{and} \quad |\bar{L}''| = |\frac{L''}{2\Lambda}| \leq 2^s < 2^{\nu-t}.$$

Thus $L' \not\subseteq L''$. $\square$

Proposition 4.37 $\mathcal{W}_{\Pi(B_\ell)}\Pi(B_\ell) = R^\times$. *Moreover if* Π' *is a subset of* $R^\times$ *which satisfies (4.24) and* $\mathcal{W}_{\Pi'}\Pi' = R^\times$ *then*, $|\Pi'| \geq ind(R) + l + \nu$.

Proof. For the first statement see Lemma 4.32. By Lemma 4.28, Corollary 4.29 and Lemma 4.33, Π' contains at least $ind(R) + 1 + t$ short and $l - 1 + (\nu - t)$ long roots. □

$\mathbf{X = C_\ell (l \geq 3)}$

We have $R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L)$ where by Theorem I.1.28, in this case S is a lattice. Recall semilattices S_1 and S_2 as in (4.15). By Definition 4.21, $ind(R) = ind(S_2) - (\nu - t)$. Let σ_i's be as in (4.17). We define

$$\Pi(C_\ell) = \{\alpha_1, \ldots, \alpha_\ell, \delta_1 - \theta_s, \ldots, \delta_t - \theta_s, \sigma_1 - \theta_\ell, \ldots, \sigma_{ind(S_2)} - \theta_\ell\}. \tag{4.38}$$

Then $|\Pi(C_\ell)| \geq ind(R) + l + t$. From Proposition 2.9 we know that $\check{R}$ is an EARS of type B_ℓ of twist number $\nu - t$. By Lemma 4.22, we have $ind(R) = ind(\check{R})$. Let $\Pi(C_\ell)\check{} := \{\check{\alpha} \mid \alpha \in \Pi(C_\ell)\}$. Then from (2.7) we have

$$\Pi(C_\ell)\check{} = \{\check{\alpha}_1, \ldots, \check{\alpha}_\ell, \delta_1 - \theta_s, \ldots, \delta_t - \theta_s, \frac{1}{2}\sigma_1 - \frac{1}{2}\theta_\ell, \ldots, \frac{1}{2}\sigma_{ind(S_2)} - \frac{1}{2}\theta_\ell\}$$

which has the form (4.31) with respect to the basis $\check{\alpha}_1, \ldots, \check{\alpha}_\ell$ of $\check{R}$. We note that θ_s is the highest long and $\frac{1}{2}\theta_\ell$ is the highest short root of $\check{R}$ with respect to the basis $\check{\alpha}_1, \ldots, \check{\alpha}_\ell$.

Lemma 4.39 $\mathcal{W}_{\Pi(C_\ell)}\Pi(C_\ell) = R^\times$.

Proof. By Lemma 4.32, we have $\mathcal{W}_{\Pi(C_\ell)\check{}}\Pi(C_\ell)\check{} = \check{R}^\times$. Hence

$$\mathcal{W}_{\Pi(C_\ell)}\Pi(C_\ell) = (\mathcal{W}_{\Pi(C_\ell)\check{}}\Pi(C_\ell)\check{})\check{} = (\check{R}^\times)\check{} = R^\times. \square$$

Lemma 4.40 *Let* Π' *be a subset of* $R^\times$ *which satisfies (4.24) and* $\mathcal{W}_{\Pi'}\Pi' = R^\times$. *Then* Π' *contains at least* $ind(R) + l - 1$ *short roots.*

Proof. Since $\mathcal{W}_{\Pi'}\Pi' = R^\times$ we have $\mathcal{W}_{\check{\Pi}'}\check{\Pi}' = \check{R}^\times$. It follows from (4.24) that $\check{\Pi}'$ satisfies (4.24) with respect to $\check{R}$. By Proposition 2.9, $\check{R}$ has twist number $\nu - t$ and by Lemma 4.22, $ind(R) = ind(\check{R})$. By Lemma 4.33, $\check{\Pi}'$ contains at least $l - 1 + (\nu - (\nu - t)) = l - 1 + t$ long roots. Thus Π' contains at least $l - 1 + t$ short roots. □

Proposition 4.41 *$\mathcal{W}_{\Pi(C_\ell)}\Pi(C_\ell) = R^\times$. Moreover if Π' is a subset of $R^\times$ which satisfies (4.24) and $\mathcal{W}_{\Pi'}\Pi' = R^\times$, then $|\Pi'| \geq ind(R) + l + t$.*

Proof. For the first statement see Lemma 4.39. We have that $\check{R}$ is an EARS of type B_ℓ with twist numer $\nu - t$. $\check{\Pi}'$ satisfies (4.24) and $\mathcal{W}_{\check{\Pi}'}\check{\Pi}' = \check{R}^\times$, therefore Proposition 4.37 gives $|\check{\Pi}'| \geq ind(\check{R}) + l + \nu = ind(R) + l + \nu$. Since $|\Pi'| = |\check{\Pi}'|$ we are done. This finishes the proof of theorem. □

Note that by Lemma 4.5, we have

$$\mathcal{W}_{\Pi(X)} = \mathcal{W}. \tag{4.42}$$

We close this section with the following interesting fact:

Proposition 4.43 *Le Π be the set defined by (4.11). Then, $R_\Pi = R \Leftrightarrow ind(R) = 0$.*

Proof. Let X be the type of R and let $\Pi(X)$ be the set which we defined in the proof of Theorem 4.23. First let $R_\Pi = R$. Then from definition of R_Π and Theorem 4.23 we have $\mathcal{W}_\Pi\Pi = R_\Pi^\times = R^\times$. Since $\bar{\Pi}$ satisfies (4.24), Theorem 4.23 gives, $|\Pi| \geq ind(R) + l + \nu$. But $|\Pi| = l + \nu$, so $ind(R) = 0$. Now let $ind(R) = 0$. Observe that $\Pi \subseteq \Pi(X)$. Thus

$$l + \nu = |\Pi| \leq |\Pi(X)| = ind(R) + l + \nu = l + \nu.$$

So $\Pi = \Pi(X)$. Thus $R_\Pi^\times = \mathcal{W}_\Pi\Pi = \mathcal{W}_{\Pi(X)}\Pi(X) = R^\times$. Therefore $R_\Pi^\times = R^\times$ and consequently $S_\Pi = S$. Thus $R_\Pi = R$. □

5 Extended Affine Root Systems of Index Zero

Let R be as in previous sections. In order to study the structure of R, in section 3 we introduced an EARS $\tilde{R}$ so that $R \subseteq \tilde{R}$. In section 4 we introduced an EARS R_Π so that $R_\Pi \subseteq R$. In this section we investigate the relations between R_Π, R and $\tilde{R}$ through the study of the centers of the corresponding Weyl groups. By Proposition 4.43, EARS's of the form R_Π are exactly the EARS's of index zero. In Proposition 5.10 we will show that the EARS's of index zero are exactly those EARS's for which the Weyl group is generated by reflections in Π. Finally we will show in Theorem 5.17 that the Weyl groups of EARS's of index zero have a so called "a presentation by conjugation".

Let

$$\mathcal{Z}_\Pi := Cent(\mathcal{W}_\Pi)$$

$$H_\Pi = \langle r_{\alpha+\delta} r_\alpha \mid \alpha \in R_\Pi^\times, \delta \in \mathcal{V}^0, \alpha+\delta \in R_\Pi \rangle.$$

In the following two lemmas we investigate the action of $\tilde{H}$ and $\mathcal{W}_\Pi$ on the dual basis $\Lambda_1, \ldots, \Lambda_\nu$ of $\delta_1, \ldots, \delta_\nu$. In fact, in the proof of Corollary 3.25 we have seen the effect of $\tilde{H}$ on $\Lambda_1, \ldots, \Lambda_\nu$, modulo $\mathcal{V}^0$. Here we need to see more than that.

Note that by Lemma 3.15, any element h of $\tilde{H}$ can be written in the form

$$h = t_{\beta_1}^{(i_1)} \cdots t_{\beta_m}^{(i_m)}, \quad (i_j, \beta_j) \in (J_s \times Q_s) \cup (J_\ell \times Q_\ell). \tag{5.1}$$

Therefore we can define the length of h, written $l(h)$, to be zero if $h = 1$ and to be the smallest positive integer m such that h can be written in the form (5.1).

Lemma 5.2 *Let $h \in \tilde{H}$, then*

$$h(\Lambda_j) \in \Lambda_j + Q_s, \ \text{if } j \in J_s \quad \text{and} \quad h(\Lambda_j) \in \Lambda_j + \frac{1}{k}Q_\ell, \ \text{if } j \in J_\ell. \tag{5.3}$$

Proof. We use induction on the length of elements in $\tilde{H}$ to prove lemma. Let $h \in \tilde{H}$ and $l(h) = 1$. Then $h = t_\beta^{(i)}$ where $(i, \beta) \in (J_s \times kQ_s) \cup (J_\ell \times Q_\ell)$. First let $(i, \beta) \in J_\ell \times Q_\ell$. Then $\beta = \dot{\beta} + k\sigma_1 + \sigma_2$ where $\sigma_1 \in \sum_{i=1}^{t} \mathbb{Z}\delta_i$ and $\sigma_2 \in \sum_{i=t+1}^{\nu} \mathbb{Z}\delta_i$. If $j \in J_s$ then from (2.5), $(\Lambda_j, \delta_i) = 0$, so from (3.9) we have

$$t_\beta^{(i)}(\Lambda_j) \;=\; \Lambda_j - (\Lambda_j, \frac{1}{k}\delta_i)\beta + [(\Lambda_j, \beta) - \frac{1}{2}(\beta, \beta)(\Lambda_j, \frac{1}{k}\delta_i)]\frac{1}{k}\delta_i$$

$$\begin{aligned}
&= \Lambda_j + (\Lambda_j, \dot{\beta} + k\sigma_1 + \sigma_2)\frac{1}{k}\delta_i \\
&= \Lambda_j + (\Lambda_j, k\sigma_1)\frac{1}{k}\delta_i \\
&= \Lambda_j + (\Lambda_j, \sigma_1)\delta_i \in \Lambda_j + Q_s.
\end{aligned}$$

If $j \in J_\ell$, then since $(\beta, \beta) \in 2k\mathbf{Z}$ we have

$$\begin{aligned}
t_\beta^{(i)}(k\Lambda_j) &= k\Lambda_j - (k\Lambda_j, \frac{1}{k}\delta_i)\beta + [(k\Lambda_j, \beta) - \frac{1}{2}(\beta, \beta)(k\Lambda_j, \frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&= k\Lambda_j - (\Lambda_j, \delta_i)\beta + (\Lambda_j, \beta)\delta_i - \frac{1}{2k}(\beta, \beta)(\Lambda_j, \delta_i)\delta_i \\
&\in k\Lambda_j - \delta_{ij}\beta + \mathbf{Z}\delta_i + \mathbf{Z}\delta_i \\
&\subseteq k\Lambda_j + Q_\ell.
\end{aligned}$$

Now we consider the case in which $(i, \beta) \in J_s \times kQ_s$. We can write $\beta = k(\dot{\beta} + \delta)$ where $\dot{\beta} \in \dot{Q}_s$ and $\delta \in \sum_{i=1}^{\nu} \mathbf{Z}\delta_i$. If $j \in J_s$, then since $(\dot{\beta}, \dot{\beta}) \in 2\mathbf{Z}$ we have

$$\begin{aligned}
t_\beta^{(i)}(\Lambda_j) &= \Lambda_j - (\Lambda_j, \delta_i)(\dot{\beta} + \delta) + [(\Lambda_j, k\dot{\beta} + k\delta) - \frac{1}{2}(k\dot{\beta}, k\dot{\beta})(\Lambda_j, \frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&\in \Lambda_j - \delta_{ij}(\dot{\beta} + \delta) + (\Lambda_j, \delta)\delta_i + \mathbf{Z}\delta_i \\
&\subseteq \Lambda_j + Q_s.
\end{aligned}$$

If $j \in J_\ell$, then $(\Lambda_j, \delta_i) = 0$ and so

$$\begin{aligned}
t_\beta^{(i)}(k\Lambda_j) &= k\Lambda_j - (k\Lambda_j, \frac{1}{k}\delta_i)\beta + [k\Lambda_j, \beta) - \frac{1}{2}(\beta, \beta)(k\Lambda_j, \frac{1}{k}\delta_i)]\frac{1}{k}\delta_i \\
&= k\Lambda_j + (\Lambda_j, k\delta)\delta_i \\
&\in k\Lambda_j + (\Lambda_j, \delta)k\delta_i \in k\Lambda_j + Q_\ell.
\end{aligned}$$

Therefore (5.3) holds when $l(h) = 1$, $h \in \tilde{H}$. Now let (5.3) holds for all $h \in \tilde{H}$ with $l(h) \leq n$. If $h' \in \tilde{H}$ with $l(h') = n+1$, then $h' = t_\beta^{(i)} h$ where $l(h) \leq n$ and $(i, \beta) \in (J_s \times kQ_s) \cup (J_\ell \times Q_\ell)$. We again divide our argument into the cases in which $(i, \beta) \in J_\ell \times Q_\ell$ or $(i, \beta) \in J_s \times kQ_s$. First let $(i, \beta) \in J_\ell \times Q_\ell$. If $j \in J_s$, then by the induction hypothesis, we have $h(\Lambda_j) = \Lambda_j + \alpha$ for some $\alpha \in Q_s$. Then using the first step of the induction, (3.10) and the fact that $(\alpha, \beta) \in k\mathbf{Z}$ we get

$$\begin{aligned}
t_\beta^{(i)} h(\Lambda_j) = t_\beta^{(i)}(\Lambda_j + \alpha) &= t_\beta^{(i)}(\Lambda_j) + t_\beta^{(i)}(\alpha) \\
&\in (\Lambda_j + Q_s) + \alpha + (\alpha, \frac{1}{k}\beta)\delta_i \\
&\subseteq \Lambda_j + Q_s + \alpha + \mathbf{Z}\delta_i \subseteq \Lambda_j + Q_s.
\end{aligned}$$

If $j \in J_\ell$, then by induction hypothesis, $h(k\Lambda_j) = k\Lambda_j + \alpha$ for some $\alpha \in Q_\ell$. Since $(\alpha, \beta) \in k\mathbf{Z}$ we get

$$\begin{aligned} t_\beta^{(i)} h(k\Lambda_j) = t_\beta^{(i)}(k\Lambda_j) + t_\beta^{(i)}(\alpha) &\in (k\Lambda_j + Q_\ell) + [\alpha + (\alpha, \frac{1}{k}\beta)\delta_i] \\ &\subseteq k\Lambda_j + Q_\ell + Q_\ell + \mathbf{Z}\delta_i \\ &\subseteq k\Lambda_j + Q_\ell. \end{aligned}$$

Now we consider the case in which $(i, \beta) \in J_s \times kQ_s$. Then $\beta = k\beta'$ for some $\beta' \in Q_s$. If $j \in J_s$ we have $h(\Lambda_j) = \Lambda_j + \alpha$ for some $\alpha \in Q_s$, then

$$t_\beta^{(i)} h(\Lambda_j) = t_\beta^{(i)}(\Lambda_j + \alpha) \in \Lambda_j + Q_s + \alpha + (\alpha, \frac{1}{k}k\beta')\delta_i \subseteq \Lambda_j + Q_s.$$

If $j \in J_\ell$, then $h(k\Lambda_j) = k\Lambda_j + \alpha$ for some $\alpha \in Q_\ell$. Since $(\alpha, \beta') \in k\mathbf{Z}$ we get

$$\begin{aligned} t_\beta^{(i)} h(k\Lambda_j) = t_\beta^{(i)}(k\Lambda_j + \alpha) &\in k\Lambda_j + Q_\ell + \alpha + (\alpha, \beta')\delta_i \\ &\in k\Lambda_j + Q_\ell + k\mathbf{Z}\delta_i \\ &\subseteq k\Lambda_j + Q_\ell. \end{aligned}$$

This finishes the induction and the proof of the lemma. $\square$

Lemma 5.4 *Let* $w \in \mathcal{W}_\Pi$.

(a) For $1 \le j \le t$ *we have*

(i)If $X = C_\ell (l \ge 3)$, *then* $w\Lambda_j \in \Lambda_j + \dot{Q}_s + \langle S \rangle$.

(ii) If $X = B_\ell (l \ge 2)$, *then* $w\Lambda_j \in \Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle$.

(b) For $t+1 \le j \le \nu$ *we have*

(i) If $X = C_\ell (l \ge 3)$ *or* B_2, *then* $w\Lambda_j \in \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S \rangle$.

(ii) If $X = B_\ell (l \ge 3)$, *then* $w\Lambda_j \in \Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S \rangle$.

(c) If $X = A_1$ *and* $1 \le j \le \nu$, *then* $w\Lambda_j \in \Lambda_j + \mathbf{Z}\delta_j + \dot{Q} + 2\langle S \rangle$.

Proof. We use induction on the length of elements in $\mathcal{W}_\Pi$ (see Definition 4.9). For parts (a) and (b) we proceed as follows. We start by showing that if $w = r_{\alpha_h}$, $1 \le h \le l$, or $w = r_{\delta_i - \theta_s}$, $1 \le i \le t$, or $w = r_{\delta_p - \theta_\ell}$, $t+1 \le p \le \nu$, then w satisfies the conditions stated in the statement. This would take care of the case in which $l(w) = 1$. Next we show that if $w' \in \mathcal{W}_\Pi$, $l(w') \le n$ and w' satisfies the statement, then elements

$$r_{\alpha_h} w',\ r_{\delta_i - \theta_s} w' \text{ and } r_{\delta_p - \theta_\ell} w',\ 1 \le h \le l,\ 1 \le i \le t,\ t+1 \le p \le \nu$$

also satisfy the conditions of the statement. This would complete the induction steps. We begin with (a). Since $1 \leq j \leq t$ and because of the definition of the bilinear form $(\ldots)$ on $\tilde{\mathcal{V}}$ (see (2.5)) we have

$$r_{\alpha_h}(\Lambda_j) = \Lambda_j, \quad r_{\delta_i - \theta_s}(\Lambda_j) = \Lambda_j - \delta_{ij}(\delta_i - \theta_s) \quad \text{and} \quad r_{\delta_p - \theta_\ell}(\Lambda_j) = \Lambda_j. \tag{5.5}$$

(i) By (5.5), (i) holds for $l(w) = 1$. Let w' satisfy (i), then from (5.5) we have

$$\begin{aligned} r_{\alpha_h} w'(\Lambda_j) \in r_{\alpha_h}(\Lambda_j + \dot{Q}_s + \langle S \rangle) &= r_{\alpha_h}(\Lambda_j) + r_{\alpha_h}(\dot{Q}_s) + \langle S \rangle \\ &\subseteq \Lambda_j + \dot{Q}_s + \langle S \rangle. \end{aligned}$$

$$\begin{aligned} r_{\delta_i - \theta_s} w'(\Lambda_j) \in r_{\delta_i - \theta_s}(\Lambda_j + \dot{Q}_s + \langle S \rangle) &\subseteq (\Lambda_j - \delta_{ij}(\delta_i - \theta_s)) + (\dot{Q}_s + \mathbf{Z}\delta_i) + \langle S \rangle \\ &\subseteq \Lambda_j + \dot{Q}_s + \langle S \rangle \quad \text{and} \end{aligned}$$

$$\begin{aligned} r_{\delta_p - \theta_\ell} w'(\Lambda_j) \in r_{\delta_p - \theta_\ell}(\Lambda_j + \dot{Q}_s + \langle S \rangle) &\subseteq \Lambda_j + (\dot{Q}_s + \mathbf{Z}\delta_p) + \langle S \rangle \\ &\subseteq \Lambda_j + \dot{Q}_s + \langle S \rangle. \end{aligned}$$

This completes the proof of (i).

(ii) By (5.5), (ii) holds for $w \in \mathcal{W}_\Pi$ with $l(w) = 1$. Let $w' \in \mathcal{W}_\Pi$ satisfy (ii), then using (5.5) and the fact that in this case $(\alpha, \check{\beta}) \in 2\mathbf{Z}$ for any $\alpha, \beta \in \dot{R}_{sh}$, we have

$$r_{\alpha_h} w'(\Lambda_j) \in r_{\alpha_h}(\Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle) \subseteq \Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle,$$

$$\begin{aligned} r_{\delta_i - \theta_s} w'(\Lambda_j) &\in r_{\delta_i - \theta_s}(\Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle) \\ &\subseteq (\Lambda_j - \delta_{ij}(\delta_i - \theta_s)) + \mathbf{Z}\delta_j + \mathbf{Z} r_{\delta_i - \theta_s}(\dot{R}_{sh}) + \langle L \rangle \\ &\subseteq \Lambda_j - \delta_{ij}(\delta_i - \theta_s) + \mathbf{Z}\delta_j + (\dot{Q}_s + 2\mathbf{Z}\delta_i) + \langle L \rangle \\ &\subseteq \Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle, \quad \text{and} \end{aligned}$$

$$\begin{aligned} r_{\delta_p - \theta_\ell} w'(\Lambda_j) \in r_{\delta_p - \theta_\ell}(\Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle) &\subseteq \Lambda_j + \mathbf{Z}\delta_j + (\dot{Q}_s + \mathbf{Z}\delta_p) + \langle L \rangle \\ &\subseteq \Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle. \end{aligned}$$

This completes the proof of (ii). Now we start the proof of part (b). Since $t + 1 \leq j \leq \nu$ we have

$$r_{\alpha_h}(\Lambda_j) = \Lambda_j, \quad r_{\delta_i - \theta_s}(\Lambda_j) = \Lambda_j \quad \text{and} \quad r_{\delta_p - \theta_\ell}(\Lambda_j) = \Lambda_j - \frac{1}{2}\delta_{pj}(\delta_p - \theta_\ell). \tag{5.6}$$

(i) By (5.6), (i) holds for $l(w) = 1$. Now let (i) holds for w'. Then from (5.6) and the fact that in this case $(\beta, \check{\alpha}) \in 2\mathbf{Z}$ for any $\beta \in \dot{R}_{lg}$, $\alpha \in \dot{R}_{sh}$ and $(\alpha, \check{\beta}) \in 2\mathbf{Z}$ for any $\alpha, \beta \in \dot{R}_{lg}$, we have

$$r_{\alpha_h} w'(\Lambda_j) \in r_{\alpha_h}(\Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle) \subseteq \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle.$$

$$\begin{aligned} r_{\delta_i - \theta_s} w'(\Lambda_j) &\in r_{\delta_i - \theta_s}(\Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle) \\ &\subseteq \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}(\dot{Q}_\ell + 2\mathbf{Z}\delta_i) + \langle S\rangle \\ &\subseteq \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle, \quad \text{and} \end{aligned}$$

$$\begin{aligned} r_{\delta_p - \theta_\ell} w'(\Lambda_j) &\in r_{\delta_p - \theta_\ell}(\Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle) \\ &\subseteq \Lambda_j - \frac{1}{2}\delta_{pj}(\delta_p - \theta_\ell) + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}(\dot{Q}_\ell + 2\mathbf{Z}\delta_p) + \langle S\rangle \\ &\subseteq \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle. \end{aligned}$$

(ii) By (5.6), (ii) holds for $l(w) = 1$. Let w' satisfy (ii). Then from (5.6) we have

$$r_{\alpha_h} w'(\Lambda_j) \in r_{\alpha_h}(\Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle) \subseteq \Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle.$$

$$\begin{aligned} r_{\delta_i - \theta_s} w'(\Lambda_j) \in r_{\delta_i - \theta_s}(\Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle) &\subseteq \Lambda_j + \frac{1}{2}(\dot{Q}_\ell + \mathbf{Z}\delta_i) + \frac{1}{2}\langle S\rangle \\ &\subseteq \Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle \quad \text{and} \end{aligned}$$

$$\begin{aligned} r_{\delta_p - \theta_\ell} w'(\Lambda_j) &\in r_{\delta_p - \theta_\ell}(\Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle) \\ &\subseteq \Lambda_j - \frac{1}{2}\delta_{pj}(\delta_p - \theta_\ell) + \frac{1}{2}(\dot{Q}_\ell + \mathbf{Z}\delta_p) + \frac{1}{2}\langle S\rangle \\ &\subseteq \Lambda_j + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\langle S\rangle. \end{aligned}$$

This finishes the proof of part (b).

(c) Let $l(w) = 1$. Then $w = r_\theta$ or $w = r_{\delta_i - \theta}$, for some $1 \leq i \leq \nu$. Then

$$r_\theta(\Lambda_j) = \Lambda_j \quad \text{and} \quad r_{\delta_i - \theta}(\Lambda_j) = \Lambda_j - \delta_{ij}(\delta_i - \theta). \tag{5.7}$$

So (c) holds for $l(w) = 1$. Now let $w' \in \mathcal{W}_\Pi$ satisfy (c). Then by (5.7),

$$r_\theta w'(\Lambda_j) \in r_\theta(\Lambda_j + \mathbf{Z}\delta_j + \dot{Q} + 2\langle S\rangle) \subseteq \Lambda_j + \mathbf{Z}\delta_j + \dot{Q} + 2\langle S\rangle \quad \text{and}$$

$$\begin{aligned} r_{\delta_i-\theta}w'(\Lambda_j) &\in r_{\delta_i-\theta}(\Lambda_j\mathbf{Z}\delta_j+\dot{Q}+2\langle S\rangle) \\ &\subseteq (\Lambda_j+\delta_{ij}(\delta_i-\theta))\mathbf{Z}\delta_j+(\dot{Q}+2\mathbf{Z}\delta_i)+2\langle S\rangle \\ &\subseteq \Lambda_j+\mathbf{Z}\delta_j+\dot{Q}+2\langle S\rangle. \end{aligned}$$

This completes the proof of the lemma. □

Proposition 5.8 *(i) For all types*

$$\tilde{\mathcal{Z}} = \langle c_{ij}^k, c_{p,q} \mid (i,j) \in J_s \times J : p,q \in J_\ell \rangle.$$

(ii) If $X = A_1$ or $C_\ell (l \geq 3)$, then

$$\mathcal{Z}_\Pi = \langle c_{i,j}^2 \mid i,j \in J, i < j \rangle.$$

(iii) If $X = B_\ell (l \geq 3)$, then

$$\mathcal{Z}_\Pi = \langle c_{i,j}^4, c_{p,q}^2, c_{s,u} \mid i,j \in J_s ; (p,q) \in J_s \times J_\ell ; s,u \in J_\ell \rangle.$$

(iv) If $X = B_2$, then

$$\mathcal{Z}_\Pi = \langle c_{i,j}^4, c_{p,q}^2 \mid i,j \in J_s ; (p,q) \in J \times J_\ell \rangle.$$

(v) For the remaining types we have

$$\mathcal{Z}_\Pi = \tilde{\mathcal{Z}}.$$

Proof. (i) Let us denote by M the right hand side of the equation in the statement. We first show that $M \subseteq \tilde{\mathcal{Z}}$. Let $(i,j) \in J_s \times J$. Then $\theta_s + \delta_i + \delta_j$ and $\theta_s + \delta_i$ are in $\tilde{R}_{sh}$. So

$$t_{k\theta_s}^{(j)} = r_{\theta_s+\delta_j} r_{\theta_s} \in \tilde{H} \quad \text{and} \quad t_{k\theta_s}^{(j)} c_{ij}^k = t_{k\theta_s}^{(j)} t_{k\delta_i}^{(j)} = t_{k(\theta_s+\delta_i)}^{(j)} = r_{\theta_s+\delta_i+\delta_j} r_{\theta_s+\delta_i} \in \tilde{H}.$$

From these equalities we get $c_{ij}^k \in \tilde{H}$. So $c_{ij}^k \in \tilde{H} \cap C = \tilde{\mathcal{Z}}$. Now if $p,q \in J_\ell$, then $\theta_\ell + \delta_p + \delta_q$ and $\theta_\ell + \delta_p$ are in $\tilde{R}_{lg}$. So

$$t_{\theta_\ell}^{(q)} = r_{\theta_\ell+\delta_q} r_{\theta_\ell} \in \tilde{H} \quad \text{and} \quad t_{\theta_\ell}^{(q)} c_{pq} = t_{\theta_\ell}^{(q)} t_{\delta_p}^{(q)} = t_{\theta_\ell+\delta_p}^{(q)} = r_{\theta_\ell+\delta_p+\delta_q} r_{\theta_\ell+\delta_p} \in \tilde{H}.$$

Thus $c_{pq} \in \tilde{H} \cap C = \tilde{\mathcal{Z}}$. Therefore $M \subseteq \tilde{\mathcal{Z}}$. To prove $M = \tilde{\mathcal{Z}}$, we must show that if $c := \prod_{(i,j) \in J_s \times J} c_{ij}^{m_{ij}} \in \tilde{\mathcal{Z}}$, then $k | m_{ij}$ for all i,j. This is because $C = \langle c_{ij} \mid i,j \in J, i < j \rangle$

is a free abelian group of rank $\nu(\nu-1)/2$ (see Lemma 3.21) and $M \leq \tilde{\mathcal{Z}} \leq C$. Note from (3.18) that for $i_0 \in J$ we have

$$\prod_{i<j} c_{ij}^{m_{ij}}(\Lambda_{i_0}) = \Lambda_{i_0} - \sum_{i<i_0} \frac{m_{ii_0}}{k}\delta_j + \sum_{i_0<j} \frac{m_{i_0j}}{k}\delta_j. \tag{5.9}$$

Now let c be as above. Fix $i_0 \in J_s$. By Lemma 5.2,

$$c(\Lambda_{i_0}) \in \Lambda_{i_0} + Q_s.$$

This and (5.9) gives $k|m_{ii_0}$ for all $i < i_0$ and $k|m_{i_0j}$ for all $i_0 < j$. Since i_0 was arbitrary, we get $k|m_{ij}$ for all m_{ij}, $(i,j) \in J_s \times J$. This completes the proof of part (i).

(ii) First let $X = A_1$. We have $\mathcal{Z}_\Pi \subseteq C = \langle c_{ij} \mid i,j \in J, i<j\rangle$. By Lemma 3.21(iv), we have $(t_\theta^{(i)}, t_\theta^{(i)}) = c_{ij}^{(\theta,\theta)} = c_{ij}^2$, so $c_{ij}^2 \in \mathcal{Z}_\Pi$, for any $i,j \in J$. Now let $c := \prod_{i<j} c_{ij}^{m_{ij}} \in \mathcal{Z}_\Pi \subseteq \mathcal{W}_\Pi$, $m_{ij} \in \mathbf{Z}$. Fix $i_0 \in J$. By Lemma 5.4(c), we have

$$c(\Lambda_{i_0}) \in \Lambda_{i_0} + \mathbf{Z}\delta_{i_0} + \dot{Q} + 2\langle S\rangle.$$

This and (5.9) gives $2|m_{i_0i}$, $i < i_0$ and $2|m_{i_0j}$ for $i_0 < j$. Since i_0 was arbitrary, we get $2|m_{ij}$ for all i,j. This takes care of the case $X = A_1$. Now let $X = C_\ell (l \geq 3)$. We know that there exist roots $\alpha, \beta, \gamma \in \dot{R}_{sh}$ and $\alpha' \in \dot{R}_{lg}$ such that $(\alpha,\beta) = 1$ and $(\gamma,\alpha') = 2$. Thus if $i,j \in J_s$ we have

$$(t_{k\alpha}^{(i)}, t_{k\beta}^{(j)}) = c_{ij}^{\frac{1}{k}(k\alpha,k\beta)} = c_{ij}^k = c_{ij}^2 \in \mathcal{Z}_\Pi.$$

If $i \in J_s, j \in J_\ell$, then

$$(t_{k\gamma}^{(i)}, t_{\alpha'}^{(j)}) = c_{ij}^{\frac{1}{k}(k\gamma,\alpha')} = c_{ij}^2 \in \mathcal{Z}_\Pi.$$

If $i,j \in J_\ell$, then

$$(t_{\alpha'}^{(i)}, t_{\alpha'}^{(j)}) = c_{ij}^{\frac{1}{k}(\alpha',\alpha')} = c_{ij}^2 \in \mathcal{Z}_\Pi.$$

Thus $c_{ij}^2 \in \mathcal{Z}_\Pi$ for $i,j \in J$, $i<j$. Now let $c := \prod_{i<j} c_{ij}^{m_{ij}} \in \mathcal{Z}_\Pi \subseteq \mathcal{W}_\Pi$. Fix $i_0 \in J$. From Lemma 5.4(a),(b) we have

$$c(\Lambda_{i_0}) \in \Lambda_{i_0} + \langle S\rangle \quad (\bmod \frac{1}{2}\dot{Q}_s + \frac{1}{2}\mathbf{Z}\delta_{i_0}).$$

Again using (5.9), we get $2|m_{ij}$ for all i,j. This completes the proof of (ii).

(iii) We know that in this case there exist roots $\alpha \in \dot{R}_{sh}$ and $\beta, \gamma, \gamma' \in \dot{R}_{lg}$ such that $(\alpha, \beta) = k = 2$, $(\gamma, \gamma') = 2$. Then if $i, j \in J_s$ we have

$$(t_{k\alpha}^{(i)}, t_{k\alpha}^{(j)}) = c_{ij}^{\frac{1}{k}(k\alpha, k\alpha)} = c_{ij}^{2k} = c_{ij}^{4} \in \mathcal{Z}_{\Pi}.$$

If $i \in J_s$, $j \in J_\ell$, then

$$(t_{k\alpha}^{(i)}, t_{\beta}^{(j)}) = c_{ij}^{\frac{1}{k}(k\alpha, \beta)} = c_{ij}^{2} \in \mathcal{Z}_{\Pi}.$$

If $i, j \in J_\ell$, then

$$(t_{\gamma}^{(i)}, t_{\gamma'}^{(j)}) = c_{ij}^{\frac{1}{k}(\gamma, \gamma')} = c_{ij} \in \mathcal{Z}_{\Pi}.$$

Now let $c := \prod_{(i,j) \in J_s \times J} c_{ij}^{m_{ij}} \in \mathcal{Z}_{\Pi} \subseteq \mathcal{W}_{\Pi}$. Fix $i_0 \in J$. From Lemma 5.4(a) we have

$$c(\Lambda_{i_0}) \in \Lambda_{i_0} + \mathbf{Z}\delta_{i_0} + \dot{Q}_s + \langle L \rangle \quad \text{if } i_0 \in J_s.$$

So this and (5.9) gives $4|m_{ij}$ if $i, j \in J_s$ and $2|m_{ij}$ if $(i, j) \in J_s \times J_l$. (Note that $\langle L \rangle = \sum_{i=1}^{\nu} k_i \mathbf{Z}\delta_i$, where $k_i = 2$ for $i \in J_s$ and $k_i = 1$ for $i \in J_\ell$). This completes the proof of part (iii).

(iv) In this case there exist roots $\alpha \in \dot{R}_{sh}$, $\beta \in \dot{R}_{lg}$ so that $(\alpha, \beta) = 2$. Then similarly to the previous cases we see that $c_{ij}^{4} \in \mathcal{Z}_{\Pi}$, $i, j \in J_s$ and $c_{pq}^{2} \in \mathcal{Z}_{\Pi}$, $(p, q) \in J \times J_\ell$. Now let $c := \prod_{i<j} c_{ij}^{m_{ij}} \in \mathcal{Z}_{\Pi} \subseteq \mathcal{W}_{\Pi}$. If $i_0 \in J_l$, then from Lemma 5.4(b) we have

$$c(\Lambda_{i_0}) \in \Lambda_{i_0} + \frac{1}{2}\dot{Q}_\ell + \frac{1}{2}\mathbf{Z}\delta_{i_0} + \langle S \rangle.$$

This and (5.9) gives $2|m_{ij}$ if $i, j \in J_l$. Since $c_{ij}^{2} \in \mathcal{Z}_{\Pi}$ fpr $i, j \in J_l$, we get

$$c_1 := \prod_{\substack{i<j \\ (i,j) \in J_s \times J}} c_{ij}^{m_{ij}} \in \mathcal{Z}_{\Pi}.$$

Then the same arguement as in part (iii) gives $4|m_{ij}$ for $i, j \in J_s$ and $2|m_{ij}$ for $(i, j) \in J_s \times J_l$.

(v) Now let X be of type different from A_1, B_ℓ or C_ℓ. By Proposition 4.13, we have $\mathcal{W} = \bar{\mathcal{W}} = \mathcal{W}_{\Pi}$. Thus $\mathcal{Z}_{\Pi} = \tilde{\mathcal{Z}}$. □

The following proposition gives a characterization of EARS's of index zero in terms of their Weyl groups.

Proposition 5.10 $\mathcal{W}\Pi = R^{\times} \Leftrightarrow R^{\times} = R_{\Pi}^{\times} \Leftrightarrow ind(R) = 0 \Leftrightarrow \mathcal{W} = \mathcal{W}_{\Pi}$.

Proof. From Proposition 4.43 we have $R = R_\Pi \Leftrightarrow ind(R) = 0$. By Lemma 2.21 (replacing R' with R and R with R_Π), we get $\mathcal{W}R_\Pi^\times \subseteq R_\Pi^\times$. So if $\mathcal{W}\Pi = R^\times$, then

$$R^\times = \mathcal{W}\Pi \subseteq \mathcal{W}R_\Pi^\times \subseteq R_\Pi^\times = \mathcal{W}_\Pi\Pi \subseteq \mathcal{W}R^\times \subseteq R^\times.$$

Thus $R^\times = R_\Pi^\times$. On the other hand if $R^\times = R_\Pi^\times$, then $\mathcal{W} = \mathcal{W}_\Pi$ and so $\mathcal{W}\Pi = \mathcal{W}_\Pi\Pi = R_\Pi^\times = R^\times$. So to complete the proof we only need to show $\mathcal{W} = \mathcal{W}_\Pi \iff R^\times = R_\Pi^\times$. If $X \neq A_1$, B_ℓ or C_ℓ, then by Proposition 4.13 and Corollary 4.14, we have $\mathcal{W}\Pi = R^\times$ and $\mathcal{W} = \mathcal{W}_\Pi$. So we can assume that $X = A_1$, $B_\ell(l \geq 2)$ or $C_\ell(l \geq 3)$. Let $\mathcal{W} = \mathcal{W}_\Pi$. We want to show $R^\times = R_\Pi^\times$. Thus we are done if we show that $S = S_\Pi$ and $L = L_\Pi$. Since $S_\Pi \subseteq S$ and $L_\Pi \subseteq L$ we only need to show that $S \subseteq S_\Pi$ and $L \subseteq L_\Pi$. Let $\delta \in S$ and $\lambda \in L$. We have $\langle S\rangle = \langle S_\Pi\rangle$ and $\langle L\rangle = \langle L_\Pi\rangle$. Since $S_\Pi + \langle L_\Pi\rangle \subseteq S_\Pi$ we have $\delta \in S_\Pi \Leftrightarrow \delta + \langle L_\Pi\rangle \subseteq S_\Pi \Leftrightarrow \delta + \langle L\rangle \subseteq S_\Pi$. So we may assume that

$$\delta = \epsilon_1\delta_1 + \cdots + \epsilon_t\delta_t \quad \text{where} \quad \epsilon_i \in \{0, 1\},\ 1 \leq i \leq t. \tag{5.11}$$

We can assume that there is some j, $1 \leq j \leq t$ such that $\epsilon_j \neq 0$, because otherwise $\delta = 0$ and we are done. Also since $L_\Pi + 2\langle S_\Pi\rangle \subseteq L_\Pi$ we have $\lambda \in L_\Pi \Leftrightarrow \lambda + 2\langle S_\Pi\rangle \subseteq L_\Pi \Leftrightarrow \lambda + 2\langle S\rangle \subseteq L_\Pi$. So we can assume that

$$\lambda = \epsilon_{t+1}\delta_{t+1} + \cdots + \epsilon_\nu\delta_\nu \quad \text{where} \quad \epsilon_i \in \{0, 1\},\ t+1 \leq i \leq \nu. \tag{5.12}$$

Again we can assume that there is some j, $t+1 \leq j \leq \nu$ such that $\epsilon_j \neq 0$. We consider different types of X separately and in each case we show that $\delta \in S_\Pi$ and $\lambda \in L_\Pi$(for $X = A_1$ we show $\delta \in S_\Pi$).

$X = C_\ell(l \geq 3)$. By Lemma 4.7, we have $S = \langle S\rangle = \langle S_\Pi\rangle = S_\Pi$. So $\delta \in S_\Pi$. We have $\theta_\ell + \lambda \in R^\times$, so $r_{\theta_\ell+\lambda} \in \mathcal{W} = \mathcal{W}_\Pi$. By Lemma 5.4(b), we have

$$r_{\theta_\ell+\lambda}(\Lambda_j) \in \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S\rangle.$$

On the other hand, since $\epsilon_j \neq 0$,

$$r_{\theta_\ell+\lambda}(\Lambda_j) = \Lambda_j - (\Lambda_j, (\theta_\ell + \lambda)^\vee)(\theta_\ell + \lambda) = \Lambda_j - \frac{1}{2}(\theta_\ell + \lambda).$$

So $\frac{1}{2}\lambda \in \frac{1}{2}\mathbf{Z}\delta_j + \langle S\rangle$. Thus

$$\lambda \in \mathbf{Z}\delta_j + 2\langle S\rangle \subseteq L_\Pi + 2\langle S_\Pi\rangle \subseteq L_\Pi.$$

$X = B_\ell (l \geq 2)$. We have $\theta_s + \delta \in R^\times$, so $r_{\theta_s+\delta} \in \mathcal{W} = \mathcal{W}_\Pi$. By Lemma 5.4(a), we have

$$r_{\theta_s+\delta}(\Lambda_j) \in \Lambda_j + \mathbf{Z}\delta_j + \dot{Q}_s + \langle L \rangle.$$

On the other hand, since $\epsilon_j \neq 0$,

$$r_{\theta_s+\delta}(\Lambda_j) = \Lambda_j - (\Lambda_j, (\theta_s + \delta)^\vee)(\theta_s + \delta) = \Lambda_j - (\theta_s + \delta).$$

Thus

$$\delta \in \mathbf{Z}\delta_j + \langle L \rangle \subseteq S_\Pi + \langle L_\Pi \rangle \subseteq S_\Pi.$$

If $l \geq 3$, then by Lemma 4.7 we have $L = \langle L \rangle = \langle L_\Pi \rangle = L_\Pi$, so $\lambda \in L_\Pi$. For $X = B_2$, we have $\theta_\ell + \lambda \in R^\times$, so $r_{\theta_\ell+\lambda} \in \mathcal{W} = \mathcal{W}_\Pi$. Then by Lemma 5.4(b) we have

$$r_{\theta_\ell+\lambda}(\Lambda_j) \in \Lambda_j + \frac{1}{2}\mathbf{Z}\delta_j + \frac{1}{2}\dot{Q}_\ell + \langle S \rangle.$$

On the other hand

$$r_{\theta_\ell+\lambda}(\Lambda_j) = \Lambda_j - \frac{1}{2}(\theta_\ell + \lambda).$$

Thus

$$\lambda \in \mathbf{Z}\delta_j + 2\langle S \rangle \subseteq L_\Pi + 2\langle S_\Pi \rangle \subseteq L_\Pi.$$

$X = A_1$. By Lemma 5.4(c), we have

$$r_{\theta+\delta}(\Lambda_j) \in \Lambda_j + \mathbf{Z}\delta_j + \dot{Q} + 2\langle S \rangle.$$

On the other hand

$$r_{\theta+\delta}(\Lambda_j) = \Lambda_j - (\theta + \delta) \text{ and so } \delta \in \mathbf{Z}\delta_j + 2\langle S \rangle \subseteq S_\Pi + 2\langle S_\Pi \rangle = S_\Pi.$$

□

Proposition 5.13 *(i)* $H_\Pi = \mathcal{Z}_\Pi(\prod_{i \in J_s} H_s^{(i)})(\prod_{j \in J_\ell} H_\ell^{(j)})$,
(ii) $(H, H) = (\tilde{H}, \tilde{H}) = (H_\Pi, H_\Pi) = \mathcal{Z}_\Pi$,
(iii) $\mathcal{W} = \mathcal{W}_\Pi \Leftrightarrow \mathcal{Z} = \mathcal{Z}_\Pi \Leftrightarrow (H, H) = Cent(\mathcal{W})$.

Proof. By Proposition 4.13, R_Π is an EARS of the same type and the same twist number of R and by 4.12, $\langle S_\Pi \rangle = \langle S \rangle$ and $\langle L_\Pi \rangle = \langle L \rangle$. Thus we can apply Corollary 3.23 to R_Π in place of R. This gives (i).

(ii) Applying Proposition 3.28 to H_Π in place of H we get

$$(\tilde{H}, \tilde{H}) = (H_\Pi, H_\Pi) \subseteq \mathcal{Z}_\Pi.$$

So it remains to show that $\mathcal{Z}_\Pi \subseteq (H_\Pi, H_\Pi)$. But we have already seen this in the proof of Proposition 5.8 for the types A_1, B_ℓ and C_ℓ. Now if X is simply laced of $rank > 1$, then from Proposition 5.8(iv) we have $\mathcal{Z}_\Pi = \langle c_{ij} \mid i, j \in J, i < j \rangle$. Since in this case $k = 1$ and there exist roots $\alpha, \beta \in \dot{R}$ such that $(\alpha, \beta) = 1$, from Lemma 3.21(iv) it follows that $\mathcal{Z}_\Pi \subseteq (H_\Pi, H_\Pi)$. For the cases $X = F_4$ or G_2, the inclusion $\mathcal{Z}_\Pi \subseteq (H_\Pi, H_\Pi)$ follows from Proposition 5.8(iv), Lemma 3.21(iv) and the facts that there are roots $\alpha, \beta \in \dot{R}_{sh}$ such that $(\alpha, \beta) = 1$, roots $\alpha \in \dot{R}_{sh}$ and $\beta \in \dot{R}_{lg}$ such that $(\alpha, \beta) = k$ and roots $\alpha, \beta \in \dot{R}_{lg}$ such that $(\alpha, \beta) = k$.

(iii) The assertion $\mathcal{Z} = \mathcal{Z}_\Pi \Leftrightarrow (H, H) = Cent(\mathcal{W})$ is clear from part (ii). Now let $\mathcal{Z} = \mathcal{Z}_\Pi$. Since $\mathcal{W} = \dot{\mathcal{W}} \propto H$ and $\mathcal{W}_\Pi = \mathcal{W}_{R_\Pi} = \dot{\mathcal{W}} \propto H_\Pi$, the equality $\mathcal{W} = \mathcal{W}_\Pi$ follows from the equality $H = H_\Pi$ which holds because of Corollary 3.23, part (i) and the assumption $\mathcal{Z} = \mathcal{Z}_\Pi$. If $\mathcal{W} = \mathcal{W}_\Pi$, then clearly $\mathcal{Z} = Cent\mathcal{W} = Cent\mathcal{W}_\Pi = \mathcal{Z}_\Pi$. □

Recall that R has nullity ν and twist number t and that $R_\Pi \subseteq R \subseteq \tilde{R}$ and $\mathcal{W}_\Pi \subseteq \mathcal{W} \subseteq \tilde{\mathcal{W}}$.

Proposition 5.14

(i) If $X = A_1$, then $[\tilde{\mathcal{W}} : \mathcal{W}_\Pi] = 2^{\binom{\nu}{2}}$.

(ii) If $X = C_\ell (l \geq 3)$, then $[\tilde{\mathcal{W}} : \mathcal{W}_\Pi] = 2^{\binom{\nu-t}{2}}$.

(iii) If $X = B_\ell (l \geq 3)$, then $[\tilde{\mathcal{W}} : \mathcal{W}_\Pi] = 2^{\binom{t}{2}}$.

(iv) If $X = B_2$, then $[\tilde{\mathcal{W}}, \mathcal{W}_\Pi] = 2^{\binom{t}{2} + \binom{\nu-t}{2}}$.

Proof. By Proposition 2.22, $\mathcal{W}_\Pi \lhd \tilde{\mathcal{W}}$. By Corollary 3.23 and Proposition 5.13, $[\tilde{\mathcal{W}} : \mathcal{W}_\Pi] = [\tilde{\mathcal{Z}} : \mathcal{Z}_\Pi]$. Now (i)-(iv) follows from Proposition 5.8. □

We conclude this section by showing that an EARS of index zero has a presentation which following [Kr] (see also [Sh]), we call "a presentation by conjugation". This presentation is well-known for finite and affine Weyl groups (see [St] and [M-P, Proposition 5.3.3]). For the group which arises from the vertex representation of a toroidal Lie algebra with $\nu = 2$, this result is obtained by [Sh]. [Kr] generalized this for simply laced toroidal Weyl groups of rank > 1 with a method different from [Sh]. Using an approach similar to [Kr], we now generalize this result to all EARS of index zero. First we need to introduce some new notation. For $j \in J$ we set

$$\begin{aligned}
\mathcal{V}_j &= \dot{\mathcal{V}} \oplus \mathbf{R}\delta_j \oplus \mathbf{R}\Lambda_i,\\
Q_j &= \textstyle\sum_{i=1}^{\nu} \mathbf{Z}\alpha_i \oplus k_j\mathbf{Z}\delta_j, \quad (k_j \text{ is defined in } (3.2))\\
\Pi_j &= \{\alpha_1, \dots, \alpha_\ell, \alpha_{\ell+j}\},\\
\mathcal{W}_j &= \langle r_{\alpha_i} \mid i \in \dot{I} \cup \{l+j\}\rangle,\\
R_j &= \mathbf{Z}\delta_j \cup (\dot{R}_{sh} + \mathbf{Z}\delta_j) \cup (\dot{R}_{lg} + k_j\mathbf{Z}\delta_j).
\end{aligned} \tag{5.15}$$

The following lemma is well-known from the theory of finite and affine root systems and Weyl groups. (See [St], [Ka], [M-P].)

Lemma 5.16 *(i) R_j, $j \in J$ is an affine root system with root lattice Q_j and the group $\mathcal{W}_j$ can be identified with the affine Weyl group of R_j. Moreover R_j is of twisted type if $j \in J_s$ and is of untwisted type if $j \in J_\ell$.*

(ii) $\mathcal{W}_j = \begin{cases} \dot{\mathcal{W}} \propto H_s^{(i)} \cong \dot{\mathcal{W}} \propto k\dot{Q}_s, & \text{if} j \in J_s, \\ \dot{\mathcal{W}} \propto H_\ell^{(j)} \cong \dot{\mathcal{W}} \propto \dot{Q}_\ell, & \text{if} j \in J_\ell. \end{cases}$ *(Canonically.)*

(iii) $R_j^\times = \mathcal{W}_j\Pi_j$ for $j \in J$.

(iv) $\dot{W}$ is the group defined by generators r_α, $\alpha \in \dot{R}^\times$ and the relations

$$r_\alpha^2 = 1, \ \alpha \in \dot{R}^\times \quad \textit{and} \quad r_\alpha r_\beta r_\alpha = r_{r_\alpha(\beta)}, \ \alpha, \beta \in \dot{R}^\times.$$

(v) $\mathcal{W}_j$ is the group defined by generators r_α, $\alpha \in R_j^\times$ and the relations

$$r_\alpha^2 = 1, \ \alpha \in R_j^\times \quad \textit{and} \quad r_\alpha r_\beta r_\alpha = r_{r_\alpha(\beta)}, \ \alpha, \beta \in R_j^\times.$$

Theorem 5.17 *Let $ind(R) = 0$. Let $\hat{\mathcal{W}}$ be the group defined by generators $\hat{r}_\alpha$, $\alpha \in R^\times$ and the relations*

$$(i) \ \hat{r}_\alpha^2 = 1, \ \alpha \in R^\times \quad \textit{and} \quad (ii) \ \hat{r}_\alpha \hat{r}_\beta \hat{r}_\alpha^{-1} = \hat{r}_{r_\alpha(\beta)}, \ \alpha, \beta \in R^\times.$$

Then $\mathcal{W} \cong \hat{\mathcal{W}}$.

Proof. By Lemma 2.23 and the fact that $r_\alpha^2 = 1$ for $\alpha \in R^\times$, it is clear that the assignment $\hat{r}_\alpha \longmapsto r_\alpha$ induces a unique epimorphism ψ from $\hat{\mathcal{W}}$ onto $\mathcal{W}$. By the definition of $\hat{\mathcal{W}}$, we have

$$\hat{w}\hat{r}_\alpha\hat{w}^{-1} = \hat{r}_{\psi(\hat{w})\alpha}, \quad \hat{w} \in \hat{\mathcal{W}}, \alpha \in R^\times. \tag{5.18}$$

By Lemma 5.16(iv)-(v) the restriction of ψ to the subgroups $\dot{\hat{\mathcal{W}}} := \langle \hat{r}_\alpha \mid \alpha \in \dot{R}^\times \rangle$ and $\dot{\hat{\mathcal{W}}}_j := \langle \hat{r}_\alpha \mid \alpha \in R_j^\times \rangle$, $j \in J$, of $\hat{\mathcal{W}}$ induces isomorphisms $\dot{\psi} : \dot{\hat{\mathcal{W}}} \longrightarrow \dot{\mathcal{W}}$ and $\psi_j : \hat{\mathcal{W}}_j \longrightarrow \mathcal{W}_j$, $j \in J$. We define

$$\hat{t}_\alpha^{(j)} := \psi_j^{-1}(t_\alpha^{(j)}) \quad \text{for} \quad (j, \alpha) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell). \tag{5.19}$$

Then for $i \in J_s$ and $p \in J_\ell$, using the isomorphisms ψ_i and ψ_p, we get

$$\hat{t}_{k\alpha}^{(i)} \hat{t}_{k\beta}^{(i)} = \hat{t}_{k(\alpha+\beta)}^{(i)}, \ \alpha, \beta \in \dot{Q}_s, \tag{5.20}$$

$$\hat{t}_\alpha^{(p)} \hat{t}_\beta^{(p)} = \hat{t}_{\alpha+\beta}^{(p)}, \ \alpha, \beta \in \dot{Q}_\ell, \tag{5.21}$$

$$\hat{\dot{w}} \hat{t}_{k\alpha}^{(i)} \hat{\dot{w}}^{-1} = \hat{t}_{k\dot{w}\alpha}^{(i)}, \ \alpha \in \dot{Q}_s, \dot{w} \in \dot{\mathcal{W}}, \tag{5.22}$$

$$\hat{\dot{w}} \hat{t}_\alpha^{(p)} \hat{\dot{w}}^{-1} = \hat{t}_{\dot{w}\alpha}^{(p)}, \ \alpha \in \dot{Q}_\ell, \ \dot{w} \in \dot{\mathcal{W}}. \tag{5.23}$$

Let

$$\begin{aligned} \hat{H}_s^{(i)} &= \langle \hat{t}_\alpha^{(i)} \mid \alpha \in k\dot{Q}_s \rangle \leq \hat{\mathcal{W}}, \quad (i \in J_s) \quad \text{and} \\ \hat{H}_\ell^{(p)} &= \langle \hat{t}_\alpha^{(p)} \mid \alpha \in \dot{Q}_\ell \rangle \leq \hat{\mathcal{W}} \quad (p \in J_\ell). \end{aligned}$$

To proceed with the proof of the theorem we need a few lemmas.

Lemma 5.24 *$\hat{\mathcal{W}}$ is generated by the subgroups $\dot{\hat{\mathcal{W}}}$, $\hat{H}_s^{(i)}, i \in J_s$ and $\hat{H}_\ell^{(p)}, p \in J_\ell$.*

Proof. Let $\alpha \in R^\times$. By Proposition 5.10, there exists $\alpha_i \in \Pi$ and an element $w = r_{\beta_1} \cdots r_{\beta_m}$, $\beta_j \in \Pi$ such that $w\alpha_i = \alpha$. Let $\hat{w} = \hat{r}_{\beta_1} \cdots \hat{r}_{\beta_m} \in \hat{\mathcal{W}}$. Then by (5.18) we have

$$\hat{w} \hat{r}_{\alpha_i} \hat{w}^{-1} = \hat{r}_{\psi(\hat{w})\alpha_i} = \hat{r}_{w\alpha_i} = \hat{r}_\alpha.$$

Therefore $\hat{\mathcal{W}} = \langle \hat{r}_{\alpha_i} \mid \alpha_i \in \Pi \rangle$. Now for $i \in J_s$ and $p \in J_\ell$ we have (see (3.7),(3.8) and (3.9))

$$r_{\alpha_{\ell+i}} = r_{\delta_i - \theta_s} = t_{-k\theta_s}^{(i)} r_{\theta_s} \quad \text{and} \quad r_{\alpha_{\ell+p}} = r_{\delta_p - \theta_\ell} = t_{-\theta_\ell}^{(p)} r_{\theta_\ell}.$$

Using the isomorphisms ψ_i and ψ_p we get

$$\hat{r}_{\alpha_{\ell+i}} = \hat{t}_{-k\theta_s}^{(i)} \hat{r}_{\theta_s} \quad \text{and} \quad \hat{r}_{\alpha_{\ell+p}} = \hat{t}_{-\theta_\ell}^{(p)} \hat{r}_{\theta_\ell}.$$

Thus all generators of $\hat{\mathcal{W}}$ belong to the subgroup generated by $\dot{\hat{\mathcal{W}}}$, $\hat{H}_s^{(i)}, i \in J_s$ and $\hat{H}_\ell^{(p)}, p \in J_\ell$. This completes the proof of the lemma. □

Lemma 5.25 *For $(i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$ we have*

$$(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta) \in Cent(\hat{\mathcal{W}}).$$

where (x,y) denotes the group commutator $xyx^{-1}y^{-1}$.

Proof. Let $\gamma \in R^\times$. Then from (5.18), (5.19) and Lemma 3.21(iv) we have

$$(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta)\hat{r}_\gamma(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta)^{-1} = \hat{r}_{\psi((\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta))\gamma} = \hat{r}_{(t^{(i)}_\alpha, t^{(j)}_\beta)\gamma} = \hat{r}_{c^{\frac{1}{k}(\alpha,\beta)}_{ij}\gamma} = \hat{r}_\gamma,$$

where the last equality follows from the fact that c_{ij} fixes R pointwise (see (3.18)). □

Lemma 5.26 *Let $(i,\alpha),(j,\beta) \in (J_s \times k\dot{R}_{sh}) \cup (J_\ell \times \dot{R}_{lg})$. The element $(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta)$ of $\hat{\mathcal{W}}$ is uniquely determined by the ordered pair (i,j) and the real number (α,β).*

Proof. We prove the lemma for each of the following cases separately.

(1) $(i,\alpha),(j,\beta) \in J_s \times k\dot{R}_{sh}$,

(2) $(i,\alpha),(j,\beta) \in J_\ell \times \dot{R}_{lg}$,

(3) $(i,\alpha) \in J_s \times k\dot{R}_{sh}$ and $(j,\beta) \in J_\ell \times \dot{R}_{lg}$.

Since $(\hat{t}^{(j)}_\beta, \hat{t}^{(i)}_\alpha) = (\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta)^{-1}$, the above cases covers all the possibilities.

(1) We have $\alpha = k\alpha'$ and $\beta = k\beta'$ for some $\alpha', \beta' \in \dot{R}_{sh}$. Using (5.20), (5.18), (3.8) and isomorphisms ψ_i, ψ_j we have

$$\begin{aligned}
(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta) &= (\hat{t}^{(i)}_{k\alpha'}, \hat{t}^{(j)}_{k\beta'}) \\
&= \hat{t}^{(i)}_{k\alpha'}\hat{t}^{(j)}_{k\beta'}\hat{t}^{(i)}_{-k\alpha'}\hat{t}^{(j)}_{-k\beta'} \\
&= \hat{r}_{\delta_i+\alpha'}\hat{r}_{\alpha'}\hat{r}_{\delta_j+\beta'}\hat{r}_{\beta'}\hat{r}_{\alpha'}\hat{r}_{\delta_i+\alpha'}\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'} \\
&= \hat{r}_{\delta_i+\alpha'}(\hat{r}_{\alpha'}\hat{r}_{\delta_j+\beta'}\hat{r}_{\alpha'})(\hat{r}_{\alpha'}\hat{r}_{\beta'}\hat{r}_{\alpha'})\hat{r}_{\delta_i+\alpha'}\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'} \\
&= \hat{r}_{\delta_i+\alpha'}\hat{r}_{r_{\alpha'}(\delta_j+\beta')}\hat{r}_{r_{\alpha'}(\beta')}\hat{r}_{\delta_i+\alpha'}\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'} \\
&= (\hat{r}_{\delta_i+\alpha'}\hat{r}_{r_{\alpha'}(\delta_j+\beta')}\hat{r}_{\delta_i+\alpha'})(\hat{r}_{\delta_i+\alpha'}\hat{r}_{r_{\alpha'}(\beta')}\hat{r}_{\delta_i+\alpha'})\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'}.
\end{aligned}$$

By Lemma 3.12 and (3.10), we have $r_{\delta_i+\alpha'}r_{\alpha'}(\beta') = t^{(i)}_{k\alpha'}(\beta') = \beta' + (\beta', \check{\alpha}')\delta_i$ and

$$r_{\delta_i+\alpha'}r_{\alpha'}(\delta_j+\beta') = \delta_j + r_{\delta_i+\alpha'}r_{\alpha'}(\beta') = \delta_j + \beta' + (\beta', \check{\alpha}')\delta_i.$$

Therefore

$$(\hat{t}^{(i)}_\alpha, \hat{t}^{(j)}_\beta) = \hat{r}_{\delta_j+\beta'+(\beta',\check{\alpha}')\delta_i}\hat{r}_{\beta'+(\beta',\check{\alpha}')\delta_i}\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'}.$$

Since $\beta' \in \dot{R}_{sh}$, there exists $\dot{w} \in \dot{\mathcal{W}}$ such that $\dot{w}\beta' = \theta_s$. Then

$$\begin{aligned}
(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) &= \hat{\dot{w}}(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)})\hat{\dot{w}}^{-1} \\
&= \hat{\dot{w}}\hat{r}_{\delta_j+\beta'+(\beta',\dot{\alpha}')\delta_i}\hat{r}_{\beta'+(\beta',\dot{\alpha}')\delta_i}\hat{r}_{\beta'}\hat{r}_{\delta_j+\beta'}\hat{\dot{w}}^{-1} \\
&= \hat{\dot{w}}\hat{r}_{\delta_j+\beta'+(\beta',\dot{\alpha}')\delta_i}\hat{\dot{w}}^{-1}\hat{\dot{w}}\hat{r}_{\beta'+(\beta',\dot{\alpha}')\delta_i}\hat{\dot{w}}^{-1}\hat{\dot{w}}\hat{r}_{\beta'}\hat{\dot{w}}^{-1}\hat{\dot{w}}\hat{r}_{\delta_j+\beta'}\hat{\dot{w}}^{-1} \\
&= \hat{r}_{\dot{w}(\delta_j+\beta'+(\beta',\dot{\alpha}')\delta_i)}\hat{r}_{\dot{w}(\beta'+(\beta',\dot{\alpha}')\delta_i)}\hat{r}_{\dot{w}(\beta')}\hat{r}_{\dot{w}(\delta_j+\beta')} \\
&= \hat{r}_{\delta_j+\theta_s+(\beta',\dot{\alpha}')\delta_i}\hat{r}_{\theta_s+(\beta',\dot{\alpha}')\delta_i}\hat{r}_{\theta_s}\hat{r}_{\delta_j+\theta_s},
\end{aligned}$$

where the last term is uniquely determined by the ordered pair (i,j) and the real number $(\beta', \dot{\alpha}') = (\beta', \alpha') = (\alpha, \beta)/k^2$. This completes the proof of (1).

(2) From (3.7) and isomorphisms ψ_i, ψ_j we have

$$(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) = \hat{t}_\alpha^{(i)}\hat{t}_\beta^{(j)}\hat{t}_{-\alpha}^{(i)}, \hat{t}_{-\beta}^{(j)} = \hat{r}_{\delta_i+\alpha}\hat{r}_\alpha\hat{r}_{\delta_j+\beta}\hat{r}_\beta\hat{r}_\alpha\hat{r}_{\delta_i+\alpha}\hat{r}_\beta\hat{r}_{\delta_j+\beta}.$$

Then similar to the case (1) we get

$$(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) = \hat{r}_{\delta_j+\beta+(\beta,\dot{\alpha})\delta_i}\hat{r}_{\beta+(\beta,\dot{\alpha})\delta_i}\hat{r}_\beta\hat{r}_{\delta_j+\beta}.$$

Since $\beta \in \dot{R}_{lg}$, there exists $\dot{w} \in \dot{\mathcal{W}}$ such that $w\beta = \theta_\ell$. Again using a similar argument as in the case (1) we get

$$(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) = \hat{r}_{\delta_j+\theta_\ell+(\beta,\dot{\alpha})\delta_i}\hat{r}_{\theta_\ell+(\beta,\dot{\alpha})\delta_i}\hat{r}_{\theta_\ell}\hat{r}_{\delta_j+\theta_\ell}.$$

Since $\alpha \in \dot{R}_{lg}$, then from (2.7) we have $(\beta, \dot{\alpha}) = \frac{2(\beta,\alpha)}{2k} = \frac{1}{k}(\beta, \alpha)$. This completes the proof of (2).

(3) We have $\alpha = k\alpha'$ for some $\alpha' \in \dot{R}_{sh}$. Then

$$\begin{aligned}
(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) &= (\hat{t}_{k\alpha'}^{(i)}, \hat{t}_\beta^{(j)}) \\
&= \hat{t}_{k\alpha'}^{(i)}\hat{t}_\beta^{(j)}\hat{t}_{-k\alpha'}^{(i)}\hat{t}_{-\beta}^{(j)} \\
&= \hat{r}_{\delta_i+\alpha'}\hat{r}_{\alpha'}\hat{r}_{\delta_j+\beta}\hat{r}_\beta\hat{r}_{\alpha'}\hat{r}_{\delta_i+\alpha'}\hat{r}_\beta\hat{r}_{\delta_j+\beta} \\
&= \hat{r}_{\delta_j+\beta+(\beta,\dot{\alpha}')\delta_i}\hat{r}_{\beta+(\beta,\dot{\alpha}')\delta_i}\hat{r}_\beta\hat{r}_{\delta_j+\beta}.
\end{aligned}$$

Using a similar argument as in part (2) we get

$$(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) = \hat{r}_{\delta_j+\theta_\ell+(\beta,\dot{\alpha}')\delta_i}\hat{r}_{\theta_\ell+(\beta,\dot{\alpha}')\delta_i}\hat{r}_{\theta_\ell}\hat{r}_{\delta_j+\theta_\ell}.$$

Since $\alpha' \in \dot{R}_{sh}$, we have $(\beta, \dot{\alpha'}) = (\beta, \alpha') = \frac{1}{k}(\beta, \alpha)$. This completes the proof of lemma. □

Motivated by Lemma 3.21(iv), we define

$$\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)} := (\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}), \quad (i,\alpha),(j,\beta) \in (J_s \times k\dot{R}_{sh}) \cup (J_\ell \times \dot{R}_{lg}). \tag{5.27}$$

By Lemma 5.26, $\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}$ is a well-defined element of $\hat{\mathcal{W}}$. Thus, the element

$$\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}, \quad (i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell) \tag{5.28}$$

is also a well-defined element of $\hat{\mathcal{W}}$.

Corollary 5.29 *For $(i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$ we have*

$$\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)} = (\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}).$$

Proof. We can write $\alpha = \sum_{s=1}^m \gamma_s$ and $\beta = \sum_{u=1}^n \beta_u$, where all γ_s's belong to $k\dot{R}_{sh}$ or all belong to $\dot{R}_{lg}$, and the same for all β_u's. Using (5.20), (5.21), Lemma 5.25, (5.27) and (3.27) we have

$$\begin{aligned}(\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)}) = (\prod_{s=1}^m \hat{t}_{\gamma_s}^{(i)}, \prod_{u=1}^n \hat{t}_{\beta_u}^{(j)}) &= \prod_{s=1}^m \prod_{u=1}^n (\hat{t}_{\gamma_s}^{(i)}, \hat{t}_{\beta_u}^{(j)}) \\ &= \prod_{s=1}^m \prod_{u=1}^n \hat{c}_{ij}^{\frac{1}{k}(\gamma_s,\beta_u)} \\ &= \hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}.\end{aligned}$$

□

Let

$$\hat{\mathcal{Z}} = \langle \hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)} \mid (i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell) \rangle \leq \hat{\mathcal{W}}.$$

Lemma 5.30 *(i) $\hat{\mathcal{Z}} \subseteq Cent(\hat{\mathcal{W}})$,*

(ii) $\hat{\mathcal{Z}}$ is a free abelian group of rank $\nu(\nu-1)/2$.

Proof. (i) follows from Lemma 5.25 immediately. Now let $(i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$. By Corollary 5.29, $\psi(\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}) = c_{ij}^{\frac{1}{k}(\alpha,\beta)}$. Since $ind(R) = 0$, from Proposition 5.13 we have $\mathcal{Z} = (H, H)$ where $\mathcal{Z}$ is the center of $\mathcal{W}$ and H is the subgroup of $\mathcal{W}$ defined by (3.13). Therefore, using Lemmas 3.15 and 3.21(iv) we see that elements $c_{ij}^{\frac{1}{k}(\alpha,\beta)} = (t_\alpha^{(i)}, t_\beta^{(j)})$, $(i,\alpha),(j,\beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$ generate $\mathcal{Z}$. Thus $\psi(\hat{\mathcal{Z}}) = \mathcal{Z}$ and $\mathcal{Z} = \langle c_{ij}^{\frac{1}{k}m_{ij}} \mid i < j \rangle$, where m_{ij} is the smallest positive integer so that $m_{ij} = (\alpha_i, \beta_j)$ for some $(i, \alpha_i), (j, \beta_j) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$. We claim that $\hat{\mathcal{Z}} = \langle \hat{c}_{ij}^{\frac{1}{k}m_{ij}} \mid i < j \rangle$. Indeed, from the definition of $\hat{\mathcal{Z}}$ we have $\langle \hat{c}_{ij}^{\frac{1}{k}m_{ij}} \mid i < j \rangle \subseteq \hat{\mathcal{Z}}$. On the other hand, if $\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}$, $(i, \alpha), (j, \beta) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$, is a typical generator of $\hat{\mathcal{Z}}$, then $\psi(\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)}) = c_{ij}^{\frac{1}{k}(\alpha,\beta)} \in \mathcal{Z}$, so $(\alpha, \beta) = mm_{ij}$ for some $m \in \mathbf{Z}$. Thus $\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)} = (\hat{c}_{ij}^{\frac{1}{k}m_{ij}})^m$. Now this and the fact that

$$\hat{c}_{ji}^{\frac{1}{k}(\alpha,\beta)} = (\hat{t}_\beta^{(j)}, \hat{t}_\alpha^{(i)}) = (\hat{t}_\alpha^{(i)}, \hat{t}_\beta^{(j)})^{-1} = (\hat{c}_{ij}^{\frac{1}{k}(\alpha,\beta)})^{-1},$$

gives the claim. By part (i), $\hat{\mathcal{Z}}$ is abelian. To show $\hat{\mathcal{Z}}$ is free abelian let

$$\prod_{i<j} \hat{c}_{ij}^{\frac{1}{k}n_{ij}} = 1, \quad \text{where} \quad n_{ij} \in \mathbf{Z} \text{ and } \hat{c}_{ij}^{n_{ij}} \in \hat{\mathcal{Z}}.$$

Then $c_{ij}^{\frac{1}{k}n_{ij}} = \psi(\hat{c}_{ij}^{\frac{1}{k}n_{ij}}) = 1$ in $\mathcal{Z}$. But $\mathcal{Z}$ is a free abelian group on generators $c_{ij}^{\frac{1}{k}m_{ij}}$. So, $n_{ij} = 0$ for all $i, j \in J$, $i < j$. Thus $\hat{\mathcal{Z}}$ is a free abelian group of rank $\nu(\nu - 1)/2$. □

We are now ready to complete the proof of theorem. We are done if we show that ψ is one to one. So let $\hat{w} \in \hat{\mathcal{W}}$ and $\psi(\hat{w}) = 1$. Using Lemmas 5.24, 5.25 and 5.30 we can write

$$\hat{w} = \hat{\dot{w}}(\prod_{i=1}^{t} \hat{t}_{k\gamma_i}^{(i)})(\prod_{j=t+1}^{\nu} \hat{t}_{\beta_j}^{(j)})(\prod_{i<j} \hat{c}_{ij}^{\frac{1}{k}n_{ij}})$$

where $\dot{w} \in \dot{\mathcal{W}}$, $\gamma_i \in \dot{Q}_s$, $\beta_j \in \dot{Q}_\ell$ and $n_{ij} \in \mathbf{Z}$. Then

$$1 = \psi(\hat{w}) = \dot{w}(\prod_{i=1}^{t} t_{k\gamma_i}^{(i)})(\prod_{j=t+1}^{\nu} t_{\beta_j}^{(j)})(\prod_{i<j} c_{ij}^{\frac{1}{k}n_{ij}}).$$

From Corollary 3.25, we get $\dot{w} = 1$, $\gamma_i = 0$ for $i \in J_s$, $\beta_j = 0$ for $j \in J_\ell$ and $n_{ij} = 0$ for $i, j \in J$, $i < j$. Thus $\hat{w} = 1$. This finishes the proof of theorem. □

See also [Sa-T] for another representation of EAWG's when $\nu = 2$. The representation given in [Sa-T] is a generalization of a Coxeter system. Namely the generators are in one to one correspondence with the vertices of a unique diagram attached to the root system and the relations consists of two groups: (i) generalized Coxeter relations attached to the diagram, and (ii) a power of a transformation, called Coxeter transformation, attached to the diagram.

6 Relations to Weyl Groups of Indefinite Type

Let, as before, R be an EARS of nullity ν and twist number t and let k be the integer we defined in Section 1. We keep the notation as in previous sections. Recall that

$$\dot{I} = \{i \mid 1 \leq i \leq l\}, \quad I = \{i \mid 1 \leq i \leq l + \nu\}, \quad J = \{i \mid 1 \leq i \leq \nu\},$$
$$J_s = \{1, \ldots, t\} \quad \text{and} \quad J_\ell = J \setminus J_s,$$

$\dot{A}$, the Cartan matrix of $\dot{R}$ with respect to $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ and

$\Pi = \{\alpha_i \mid i \in I\}$ where $\alpha_{\ell+j} = \delta_j - \theta_s$ if $j \in J_s$ and $\alpha_{l+j} = \delta_j - \theta_\ell$ if $j \in J_\ell$.

Let $A = (a_{ij})_{i,j \in I}$ the matrix in which $a_{i,j} := (\alpha_i, \check{\alpha}_j)$, $i, j \in I$. Let $i \in \dot{I}$, then

$$\text{if } j \in J_s, \quad (\alpha_i, \alpha_{\ell+j}) = (\alpha_i, \delta_j - \theta_s) = -(\alpha_i, \theta_s) \leq 0 \text{ and}$$
$$\text{if } j \in J_\ell, \quad (\alpha_i, \alpha_{\ell+j}) = (\alpha_i, \delta_j - \theta_\ell) = -(\alpha_i, \theta_\ell) \leq 0.$$

Thus

$$a_{i,j+l} \leq 0 \ \text{ for } i \in \dot{I}, j \in J \quad \text{and} \quad a_{\ell+i,j} \leq 0 \ \text{ for } i \in J, j \in \dot{I}. \tag{6.1}$$

Moreover

$$(\delta_i - \theta_s, (\delta_j - \theta_\ell)\check{}\,) = (\theta_s, \check{\theta}_\ell) = 1,$$
$$(\delta_i - \theta_\ell, (\delta_j - \theta_s)\check{}\,) = (\theta_\ell, \check{\theta}_s) = k \quad \text{and}$$
$$(\delta_i - \theta_s, (\delta_j - \theta_s)\check{}\,) = (\theta_s, \check{\theta}_s) = 2 = (\theta_\ell, \check{\theta}_\ell) = (\delta_i - \theta_\ell, (\delta_j - \theta_\ell)\check{}\,).$$

So

$$a_{\ell+i, l+j} = \begin{cases} 1 & \text{if } (i,j) \in J_s \times J_\ell, \\ k & \text{if } (i,j) \in J_\ell \times J_s, \\ 2 & \text{otherwise.} \end{cases} \tag{6.2}$$

Therefore A is a generalized intersection matrix of Slodowy [Sl]. By changing the sign of each positive off-diagonal enteries of A we get a generalized Cartan matrix $\hat{A}$,

$$\hat{A} = \left[\begin{array}{ccccccccc} \dot{A} & & (a_{i,\ell+j})_{i \in I, j \in J_s} & & & & (a_{i,l+j})_{i \in I, j \in J_\ell} & & \\ & 2 & -2 & \cdots & -2 & -1 & -1 & \cdots & -1 \\ & -2 & 2 & \cdots & -2 & -1 & -1 & \cdots & -1 \\ (a_{\ell+i,j})_{i \in J_s, j \in I} & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & -2 & -2 & \cdots & 2 & -1 & -1 & \cdots & -1 \\ & -k & -k & \cdots & -k & 2 & -2 & \cdots & -2 \\ & -k & -k & \cdots & -k & -2 & 2 & \cdots & -2 \\ (a_{\ell+i,j})_{i \in J_\ell, j \in I} & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & -k & -k & \cdots & -k & -2 & -2 & \cdots & 2 \end{array}\right] \tag{6.3}$$

Let

$$\hat{\Pi} = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell, \hat{\alpha}_{\ell+1}, \ldots, \hat{\alpha}_{\ell+\nu}\}$$

be a set of simple roots for the real root system $\hat{R}$ of $\hat{A}$. We can identify $\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell$ with the basis $\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell$ of $\dot{R}$. We set

$$\hat{\Pi}_j = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell, \hat{\alpha}_{\ell+j}\}, \quad (j \in J). \tag{6.4}$$

As one can read from the matrix $\hat{A}$,

$\hat{\Pi}_j$ forms a set of simple roots for an affine root system $\hat{R}_j$ which is of twisted type if $j \in J_s$ and is of untwisted type if $j \in J_\ell$. (6.5)

Denote by $\hat{\delta}_j$ the null root relative to $\hat{\Pi}_j$. That is

$$\hat{\delta}_j = \begin{cases} \theta_s + \hat{\alpha}_{\ell+j} & \text{if } j \in J_s, \\ \theta_\ell + \hat{\alpha}_{\ell+j} & \text{if } j \in J_\ell. \end{cases} \tag{6.6}$$

We list the corresponding objects for the root systems $\hat{R}$ and $\hat{R}_j$, $j \in J$:

$\hat{Q} = \sum_{i=1}^{l+\nu} \mathbf{Z}\hat{\alpha}_i = \sum_{i=1}^{l} \mathbf{Z}\hat{\alpha}_i \oplus \sum_{i=\ell+1}^{l+\nu} \mathbf{Z}\hat{\delta}_i$, the root lattice of $\hat{A}$.

$\hat{Q}_j = \dot{Q} \oplus \mathbf{Z}\hat{\delta}_j$, $j \in J$, the affine root lattice.

$\langle\cdot|\cdot\rangle$, the symmetric bilinear form on $\hat{Q}$,

$\hat{r}_i := r_{\hat{\alpha}_i}$, the simple reflection in $\hat{\alpha}_i$, $i \in I$,

$\hat{r}_\alpha$, the reflection in $\hat{\alpha} \in \hat{R}$,

$\hat{\mathcal{W}} = \langle \hat{r}_i \mid i \in I\rangle = \langle \hat{r}_\alpha \mid \alpha \in \hat{R}\rangle$, the indefinite Weyl group,

$\hat{\mathcal{W}}_j, j \in J$, the subgroup of $\hat{\mathcal{W}}$ generated by reflections $\hat{r}_1, \ldots, \hat{r}_\ell, \hat{r}_{\ell+j}$.

We note that

$$\langle\cdot|\cdot\rangle_{|_{\dot{Q}\times\dot{Q}}} = (\cdot,\cdot). \tag{6.7}$$

From Lemma 5.16, by replacing R_j and $\mathcal{W}_j$ with $\hat{R}_j$ and $\hat{\mathcal{W}}_j$, respectively, we have

$$\hat{\mathcal{W}}_j \cong \mathcal{W}_j = \begin{cases} \dot{\mathcal{W}} \propto H_s^{(j)} \cong \dot{\mathcal{W}} \propto k\dot{Q}_s & \text{if } j \in J_s, \\ \dot{\mathcal{W}} \propto H_\ell^{(j)} \cong \dot{\mathcal{W}} \propto \dot{Q}_\ell & \text{if } j \in J_\ell. \end{cases} \tag{6.8}$$

We note that

$$\langle \hat{\delta}_i | \hat{\delta}_j \rangle < 0 \quad \text{for } i, j \in J,\ i \neq j. \tag{6.9}$$

To see this let $\theta', \theta'' \in \{\theta_s, \theta_\ell\}$. Since $\hat{\delta}_i$ and $\hat{\delta}_j$ are null roots for the affine root system $\hat{R}_i$ and $\hat{R}_j$, we have

$$0 \geq (\hat{\alpha}_{\ell+i}|\hat{\alpha}_{\ell+j}) = (\hat{\delta}_i - \theta'|\hat{\delta}_j - \theta'') = (\hat{\delta}_i|\hat{\delta}_j) + (\theta'|\theta'').$$

Therefore $(\hat{\delta}_i|\hat{\delta}_j) \leq -(\theta'|\theta'') = -(\theta', \theta'') < 0$ (see (6.7)). It is also well known that (see [Ka],[M-P])

$$\begin{array}{l} \hat{\mathcal{W}} \text{ is a Coxeter group with presentation,} \\ \text{generators: } \hat{r}_i, i \in I, \text{ and} \\ \text{relations: } \begin{cases} \hat{r}_i^2 = 1 & \text{for all } i \in I \\ (\hat{r}_i\hat{r}_j)^2 = 1 & \text{if } \hat{a}_{ij}\hat{a}_{ji} = 0 \\ (\hat{r}_i\hat{r}_j)^3 = 1 & \text{if } \hat{a}_{ij}\hat{a}_{ji} = 1 \\ (\hat{r}_i\hat{r}_j)^{2k} = 1 & \text{if } \hat{a}_{ij}\hat{a}_{ji} = k, \text{ for } k \in \{2, 3\} \end{cases} \end{array} \tag{6.10}$$

Recall the definition of k from Convention 2.1. Note that for simply laced cases for which $k = 1$, we have only the first three relation in (6.10). If R is of type B_ℓ, C_ℓ or F_4, we also have the fourth relation with $k = 2$ and if R is of type G_2, then we also have the fourth relation with $k = 3$.

Remark 6.11 *It is known that the indefinite Weyl group $\hat{\mathcal{W}}$ acts faithfully on $\hat{Q}$ (see Corollary 2 of [M-P, 5.2]). Thus we can consider $\hat{\mathcal{W}}$ as a subgroup of $Aut_{\mathbb{Z}}(\hat{Q})$. For $(i, \alpha) \in (J_s \times k\dot{Q}_s) \cup (J_l \times \dot{Q}_\ell)$ we define the linear maps $\hat{t}_\alpha^{(i)}$ on $\hat{Q}$ by*

$$\hat{t}_\alpha^{(i)}(\lambda) = \lambda - (\lambda|\frac{1}{k}\hat{\delta}_i)\alpha + [(\lambda|\alpha) - \frac{1}{2}(\alpha|\alpha)(\lambda|\frac{1}{k}\hat{\delta}_i)]\frac{1}{k}\hat{\delta}_i. \tag{6.12}$$

As in (3.7) one can see that

$$\hat{r}_{\alpha+\hat{\delta}_i}\hat{r}_\alpha = \hat{t}_{k\alpha}^{(i)} \quad \text{if } (i, \alpha) \in J_s \times \dot{R}_{sh} \text{ and}$$

$$\hat{r}_{\alpha+\hat{\delta}_i}\hat{r}_\alpha = \hat{t}_{k\alpha}^{(i)} \quad \text{if } (i, \alpha) \in J_l \times \dot{R}_{lg}.$$

So $\hat{t}_\alpha^{(i)} \in \hat{\mathcal{W}}$ for $(i, \alpha) \in (J_s \times k\dot{R}_{sh}) \cup (J_\ell \times \dot{R}_{lg})$. Using the same proof given in Lemma 3.11(i), one can see that

$$\hat{t}_\alpha^{(i)}\hat{t}_\beta^{(i)} = \hat{t}_{\alpha+\beta}^{(i)} \text{ for } i \in J_s \text{ and } \alpha, \beta \in k\dot{R}_{sh} \quad \text{or} \quad i \in J_\ell \text{ and } \alpha, \beta \in \dot{R}_{lg}.$$

Thus $\hat{t}_\alpha^{(i)} \in \hat{\mathcal{W}}$ for $(i,\alpha) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell)$. Let

$$\hat{H}_s^{(i)} = \langle \hat{t}_\alpha^{(i)} \mid \alpha \in k\dot{Q}_s \rangle,\ i \in J_s; \quad \text{and} \quad \hat{H}_\ell^{(i)} = \langle \hat{t}_\alpha^{(i)} \mid \alpha \in \dot{Q}_\ell \rangle\ i \in J_\ell.$$

Then one has (see (6.8))

$$\hat{\mathcal{W}}_i = \begin{cases} \dot{\mathcal{W}} \propto \hat{H}_s^{(i)} & \text{if } i \in J_s, \\ \dot{\mathcal{W}} \propto \hat{H}_\ell^{(i)} & \text{if } i \in J_\ell. \end{cases} \tag{6.13}$$

Lemma 6.14 *$\hat{\mathcal{W}}_i \cap \hat{\mathcal{W}}_j = \dot{\mathcal{W}}$ for $i, j \in J$, $i \neq j$.*

Proof. We have $\dot{\mathcal{W}} \subseteq \hat{\mathcal{W}}_i \cap \hat{\mathcal{W}}_j$. Now let $w \in \hat{\mathcal{W}}_i \cap \hat{\mathcal{W}}_j$. By Remark 6.11, there exist $\alpha, \beta \in \dot{Q}$ such that $\hat{t}_\alpha^{(i)} w, \hat{t}_\beta^{(j)} w \in \dot{\mathcal{W}}$. Since $\hat{\mathcal{W}}_i \hat{\delta}_i = \hat{\delta}_i$ and $\hat{\mathcal{W}}_j \hat{\delta}_j = \hat{\delta}_j$ we have

$$\begin{aligned} \hat{t}_\beta^{(j)} w \hat{\delta}_j &= \hat{\delta}_j = \hat{t}_\alpha^{(i)} w \hat{\delta}_j = \hat{t}_\alpha^{(i)} \hat{\delta}_j \\ &= \hat{\delta}_j - \langle \hat{\delta}_j | \frac{1}{k} \hat{\delta}_i \rangle \alpha + [\langle \hat{\delta}_j | \alpha \rangle - \frac{1}{2} \langle \alpha | \alpha \rangle \langle \hat{\delta}_j | \frac{1}{k} \hat{\delta}_i \rangle] \frac{1}{k} \hat{\delta}_i. \end{aligned}$$

Since $\langle \hat{\delta}_i | \hat{\delta}_j \rangle \neq 0$ we get $\alpha = 0$. Hence $w \in \dot{\mathcal{W}}$. □

By Lemma 6.14, the amalgamated free product

$$F := \hat{\mathcal{W}}_{\ell+1} *_{\dot{\mathcal{W}}} \cdots *_{\dot{\mathcal{W}}} \hat{\mathcal{W}}_{\ell+\nu} \tag{6.15}$$

exists. Let $\{1\} \cup \{h_{j_i}\}_i$ be a set of coset representatives for $\hat{\mathcal{W}}_j / \dot{\mathcal{W}}$, $j \in J$. Then each element $\hat{v} \in F$ can be expressed uniquely in the form (see [Sc, Chapter 8])

$$\begin{aligned} &\hat{v} = \dot{w} h_1 \cdots h_n \quad \text{where} \\ &n \in \mathbb{Z}_{\geq 0},\ \dot{w} \in \dot{\mathcal{W}},\ h_k \in \textstyle\bigcup_{j=1}^m (\{h_{j_i}\}_i) \quad \text{and} \\ &\text{no two consecutive } h_k \text{ lie in the same group } \hat{\mathcal{W}}_j. \end{aligned} \tag{6.16}$$

Lemma 6.17 *$F = \dot{\mathcal{W}} \propto (\hat{H}_s^{(1)} * \cdots * \hat{H}_s^{(t)} * \hat{H}_\ell^{(t+1)} * \cdots * \hat{H}_\ell^{(\nu)})$.*

Proof. Any element of F can be expressed uniquely in the form (6.16). By Remark 6.11, the coset representatives for $\hat{\mathcal{W}}_j / \dot{\mathcal{W}}$ can be taken from $\hat{H}_s^{(j)}$ or $\hat{H}_\ell^{(j)}$ depending whether $j \in J_s$ or $j \in J_\ell$. But elements of $\dot{\mathcal{W}} \propto (\hat{H}_s^{(1)} * \cdots * \hat{H}_s^{(t)} * \hat{H}_\ell^{(t+1)} * \cdots * \hat{H}_\ell^{(\nu)})$ also have the same unique expression. □

Identifying $\hat{H}_s^{(i)}$ with $k\dot{Q}_s$ and $\hat{H}_\ell^{(i)}$ with $\dot{Q}_\ell$ we get

Corollary 6.18 $F \cong \dot{W} \propto (k\dot{Q}_s * \cdots * k\dot{Q}_s * \dot{Q}_\ell * \cdots * \dot{Q}_\ell)$.

We set

$$\hat{H} := \hat{H}_s^{(1)} * \cdots \hat{H}_s^{(t)} * \hat{H}_\ell^{(t+1)} * \cdots * \hat{H}_\ell^{(\nu)}.$$

Let N be the normal subgroup of F generated by elements

$$(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k}, \quad (i,j) \in J_s \times J_\ell. \tag{6.19}$$

Note that if X is simply laced or if the twist number t is 0 or ν, then $J_s = \emptyset$ or $J_\ell = \emptyset$. Thus in these cases N is the trivial subgroup of F. In general we have

Lemma 6.20 *(i) If $X = B_\ell (l \geq 2)$, $C_\ell (l \geq 3)$ or F_4, then N is the normal subgroup of $\hat{H}$ generated by elements*

$$\hat{t}_{i,j} := \hat{t}^{(i)}_{2\theta_s}\hat{t}^{(j)}_{\theta_\ell-2\theta_s}\hat{t}^{(i)}_{2(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{\theta_\ell}\hat{t}^{(i)}_{-2\theta_s}\hat{t}^{(j)}_{-(\theta_\ell-2\theta_s)}\hat{t}^{(i)}_{-2(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-\theta_\ell} \quad (i,j) \in J_s \times J_\ell. \tag{6.21}$$

(ii) If $X = G_2$, then N is the normal subgroup of $\hat{H}$ generated by elements

$$\begin{aligned}\hat{t}_{i,j} := \hat{t}^{(i)}_{3\theta_s}\hat{t}^{(j)}_{\theta_\ell-3\theta_s}\hat{t}^{(i)}_{3(2\theta_s-\theta_\ell)}\hat{t}^{(j)}_{2\theta_\ell-3\theta_s}\hat{t}^{(i)}_{3(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{\theta_\ell} \\ \hat{t}^{(i)}_{-3\theta_s}\hat{t}^{(j)}_{-(\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{-3(2\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-(2\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{-3(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-\theta_\ell} \quad (i,j) \in J_s \times J_\ell.\end{aligned} \tag{6.22}$$

Proof. By definition, N is the normal subgroup of F generated by elements $(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k}$, $(i,j) \in J_s \times J_\ell$. Now let $i \in J_s$ and $j \in J_\ell$. Using Remark 6.11, (6.8), Lemmas 3.12 and 3.11, we see that

$$\begin{aligned}\hat{r}_{\ell+i} = \hat{r}_{\hat{\delta}_i-\theta_s} = \hat{t}^{(i)}_{-k\theta_s}r_{\theta_s}, \quad \hat{r}_{\ell+j} = \hat{r}_{\hat{\delta}_j-\theta_\ell} = \hat{t}^{(j)}_{-\theta_\ell}r_{\theta_\ell}. \\ \dot{w}\hat{t}^{(i)}_{k\alpha}\dot{w}^{-1} = \hat{t}^{(i)}_{k\dot{w}\alpha} \quad \text{and} \quad \dot{w}\hat{t}^{(j)}_{\beta}\dot{w}^{-1} = \hat{t}^{(j)}_{\dot{w}\beta}, \text{ for any } \alpha \in \dot{Q}_s, \beta \in \dot{Q}_\ell.\end{aligned} \tag{6.23}$$

Now in the case (i) we have $k = 2$ and in the case (ii) we have $k = 3$. In either cases one can easily check that $(r_{\theta_s}r_{\theta_\ell})^{2k} = 1$. Then using (6.23) and a straightforward computation we obtain, if $k = 2$

$$(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k} = (\hat{r}_{\ell+i}\hat{r}_{\ell+j})^4 = (r_{\theta_s}r_{\theta_\ell})^4\hat{t}_{i,j} = \hat{t}_{i,j}$$

and if $k = 3$,

$$(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k} = (\hat{r}_{\ell+i}\hat{r}_{\ell+j})^6 = (r_{\theta_s}r_{\theta_\ell})^6\hat{t}_{i,j} = \hat{t}_{i,j}.$$

Thus the elements $(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k}$ sit inside the subgroup $\hat{H}$ of F. Note that N is in fact the subgroup of F generated by elements

$$\hat{v}(\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k}\hat{v}^{-1} = \hat{v}\hat{t}_{ij}\hat{v}^{-1}, \quad \hat{v} \in F,\ (i,j) \in J_s \times J_\ell.$$

By Lemma 6.17, we can write $\hat{v} = \dot{w}\hat{t}^{(i_1)}_{\beta_1}\cdots\hat{t}^{(i_m)}_{\beta_m}$, β_i's$\in \dot{Q}$ and $i_j \in J$, $i_j \neq i_{j+1}$. Then, using (6.23), we have, for the case $k = 2$,

$$\hat{v}\hat{t}_{ij}\hat{v}^{-1} = \dot{w}\hat{t}^{(i_1)}_{\beta_1}\cdots\hat{t}^{(i_m)}_{\beta_m}\hat{t}_{ij}\hat{t}^{(i_m)}_{-\beta_m}\cdots\hat{t}^{(i_1)}_{-\beta_1}\dot{w}^{-1} = \hat{t}^{(i_1)}_{\dot{w}\beta_1}\cdots\hat{t}^{(i_m)}_{\dot{w}\beta_m}\hat{t}_{w_{ij}}\hat{t}^{(i_m)}_{-\dot{w}\beta_m}\cdots\hat{t}^{(i_1)}_{-\dot{w}\beta_1}$$

where

$$\hat{t}_{w_{ij}} = \hat{t}^{(i)}_{2\dot{w}\theta_s}\hat{t}^{(j)}_{\dot{w}(\theta_\ell-2\theta_s)}\hat{t}^{(i)}_{2\dot{w}(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{\dot{w}\theta_\ell}\hat{t}^{(i)}_{-2\dot{w}\theta_s}\hat{t}^{(j)}_{-\dot{w}(\theta_\ell-2\theta_s)}\hat{t}^{(i)}_{-2\dot{w}(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-\dot{w}\theta_\ell}.$$

and for the case $k = 3$,

$$\begin{aligned}\hat{t}_{w_{ij}} := \hat{t}^{(i)}_{3\dot{w}\theta_s}\hat{t}^{(j)}_{\dot{w}(\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{3\dot{w}(2\theta_s-\theta_\ell)}\hat{t}^{(j)}_{2\dot{w}(\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{3\dot{w}(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{\dot{w}\theta_\ell}\\ \hat{t}^{(i)}_{-3\dot{w}\theta_s}\hat{t}^{(j)}_{-\dot{w}(\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{-3\dot{w}(2\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-\dot{w}(2\theta_\ell-3\theta_s)}\hat{t}^{(i)}_{-3\dot{w}(\theta_s-\theta_\ell)}\hat{t}^{(j)}_{-\dot{w}\theta_\ell}.\end{aligned}$$

After possible cancelation of adjacent terms which are inverse of each other we see that $\hat{v}\hat{t}_{ij}\hat{v}^{-1} \in \hat{H}$. Thus $N \subseteq \hat{H}$. □

Proposition 6.24 $\hat{\mathcal{W}} \cong F/N$.

Proof. For $j \in J$ we have $\hat{\mathcal{W}}_j \subseteq \hat{\mathcal{W}}$. Therefore the inclusions $\tau_j : \hat{\mathcal{W}}_j \to \hat{\mathcal{W}}$, $j \in J$, lead to an epimorphism $f : F \to \hat{\mathcal{W}}$, so that $f|_{\hat{\mathcal{W}}_j} = \tau_j$, $j \in J$. For $(i,j) \in J_s \times J_\ell$ we have

$$f((\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k}) = (f(\hat{r}_{\ell+i})f(\hat{r}_{\ell+j}))^{2k} = (\tau_i(\hat{r}_{\ell+i})\tau_j(\hat{r}_{\ell+j}))^{2k} = (\hat{r}_{\ell+i}\hat{r}_{\ell+j})^{2k} = 1,$$

where the last equality follows from (6.10). So $N \subseteq \ker f$. Therefore we get an induced epimorphism $\bar{f} : F/N \to \hat{\mathcal{W}}$. Considering (6.19) and that $\hat{\mathcal{W}}$ is a Coxeter group with the presentation given in (6.10), we see that the assignment $\hat{r}_i \mapsto \hat{r}_iN$, $i \in I$, defines an epimorphism $\hat{\mathcal{W}} \to F/N$, which is the inverse of $\bar{f}$. Hence $\hat{\mathcal{W}} \cong F/N$. □

Recall the EAWG $\mathcal{W}_\Pi$ defined by (4.1) and recall that $\mathcal{W}_\Pi$ is a subgroup of the EAWG $\mathcal{W}$ of R.

Lemma 6.25 *The assignment $\hat{r}_i \mapsto r_i$ defines a homomorphism π from the indefinite Weyl group $\hat{\mathcal{W}}$ into the EAWG $\mathcal{W}$. Moreover $\pi(\hat{\mathcal{W}}) = \mathcal{W}_\Pi$.*

Proof. Since $\hat{\mathcal{W}}$ is a Coxeter group, we only need to show that the relations in (6.10) are satisfied in $\mathcal{W}$, replacing $\hat{r}_i$ with r_i. We note that each of the first three type of relations in (6.10) sit inside only one of the affine Weyl groups $\hat{\mathcal{W}}_j$, $j \in J$ and so it is satisfied in $\mathcal{W}$. It remains to show that each relation of the fourth type in (6.10) is also satisfied in $\mathcal{W}$. For simply laced cases there is no such relation (as $J_\ell = \emptyset$). Then the same proof as in Lemma 6.20 shows that $(r_{\ell+i}r_{\ell+j})^{2k} = t_{i,j}$ where $t_{i,j}$ is defined in the same way as $\hat{t}_{i,j}$ by removing ^'s. Now using Lemma 3.21(iv), it is easy to see that $t_{i,j} = 1$ in $\mathcal{W}$. This takes care of the first statement. For the second statement, we have

$$\pi(\hat{\mathcal{W}}) = \pi(\langle \hat{r}_i \mid i \in I\rangle) = \langle \pi(\hat{r}_i) \mid i \in I\rangle = \langle r_i \mid i \in I\rangle = \mathcal{W}_\Pi.$$

□

According to Proposition 6.24 we can assume

$$\hat{\mathcal{W}} = \frac{F}{N} = \frac{\dot{\mathcal{W}} \propto \hat{H}}{N} \cong \dot{\mathcal{W}} \propto \frac{\hat{H}}{N}. \tag{6.26}$$

Therefore any element $\hat{w} \in \hat{\mathcal{W}}$ has an expression of the form

$$\hat{w} = \dot{w}\hat{t}^{(i_1)}_{\beta_1} \cdots \hat{t}^{(i_m)}_{\beta_m} N, \quad \dot{w} \in \dot{\mathcal{W}}, \beta_j \in \dot{Q},\ i_j \in J,\ i_j \neq i_{j+1}. \tag{6.27}$$

Let $\bar{\pi}$ be given by the composition

$$\bar{\pi} : F/N \xrightarrow{\bar{f}} \hat{\mathcal{W}} \xrightarrow{\pi} \mathcal{W},$$

where $\bar{f}$ is as in the proof of 6.24. Then for $\hat{w} \in F/N$ of the form (6.27) we have

$$\bar{\pi}(\hat{w}) = \dot{w}t^{(i_1)}_{\beta_1} \cdots t^{(i_m)}_{\beta_m}.$$

Now using 3.17(ii) we get

$$\bar{\pi}(\hat{w}) = z\dot{w}t^{(i_1)}_{\alpha_{i_1}} \cdots t^{(i_n)}_{\alpha_{i_n}} \text{ where } z \in \mathcal{Z},\ \dot{w} \in \dot{\mathcal{W}}, n \in \mathbb{Z}, \text{ and } \alpha_{i_k} = \sum_{i_j = k} \beta_j, k \in J.$$

Then by Proposition 3.30 and Corollary 3.25,

$$\pi(\hat{w}) = 1 \text{ in } \mathcal{W} \text{ iff } z = 1, \dot{w} = 1 \text{ and for all } k \in J,\ \sum_{i_j = k} \beta_j = 0.$$

So

$$\ker\bar{\pi} \subseteq K^{\cdot} := \{\hat{t}_{\beta_1}^{(i_1)} \cdots \hat{t}_{\beta_m}^{(i_m)} N \in \frac{\hat{H}}{N} : \sum_{i_j=k} \beta_j = 0 \text{ for all } k \in J\}.$$

We note that $\bar{\pi}(K^{\cdot}) \subseteq \mathcal{Z}$. Recall the set $C = \langle c_{ij} \mid 1 \leq i < j \leq \nu \rangle$ defined in Section 3. Let p_{ij} $(i < j, i, j \in J)$ be the $\mathbf{Z}$−valued projection map from $\mathcal{Z}$ to $\mathbf{Z}$, taking c_{ij}^m to m. Define

$$\bar{p}_{ij} := p_{ij} \circ \bar{\pi}|_{K^{\cdot}} : K^{\cdot} \to \mathbf{Z}.$$

Now

$$\hat{h}N := \hat{t}_{\beta_1}^{(i_1)} \cdots \hat{t}_{\beta_m}^{(i_m)} N \in \ker\bar{\pi} \Leftrightarrow \bar{p}_{ij}(\hat{h}N) = 0 \text{ for all } i, j \in J, i < j.$$

So

Proposition 6.28 *The assignment $\hat{r}_i \longmapsto r_i$ defines a homomorphism π from indefinite Weyl group $\hat{\mathcal{W}}$ into the EAWG $\mathcal{W}$ with $\pi(\hat{\mathcal{W}}) = \mathcal{W}_\Pi$. Moreover*

$$ker\pi = \{\hat{h}N = \hat{t}_{\beta_1}^{(i_1)} \cdots \hat{t}_{\beta_m}^{(i_m)} N : (i_j, \beta_j) \in (J_s \times k\dot{Q}_s) \cup (J_\ell \times \dot{Q}_\ell),\ 1 \leq j \leq m,$$
$$\textstyle\sum_{i_j=k} \beta_j = 0 \textit{ for all } k \in J,:\ \bar{p}_{ij}(\hat{h}N) = 0,\ i, j \in J,\ i < j\}.$$

Remark 6.29 *Note that Proposition 6.28 generalizes Proposition 2.5 of [M-S] to the class of EAWG's. Indeed, if R is simply laced of rank > 1, then by Corollary 4.14, $\mathcal{W}_\Pi = \mathcal{W}$ and as we have already seen, N is the trivial subgroup of F in this case. So Proposition 6.27 becomes identical to Proposition 2.5 of [M-S]. Note also that if $t = 0$ or $t = \nu$, then $N = (1)$. According to Proposition 6.28, any EAWG of an EARS of index zero is homomorphic image of some Weyl group of indefinite type.*

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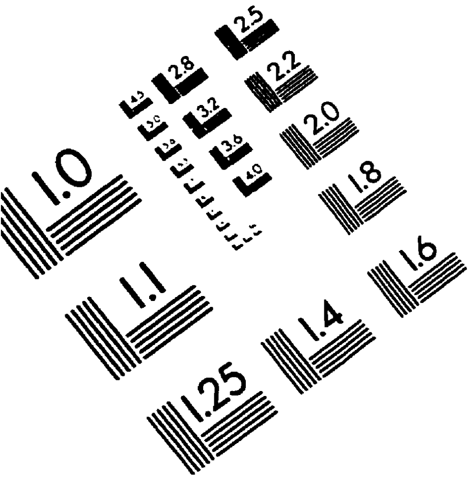
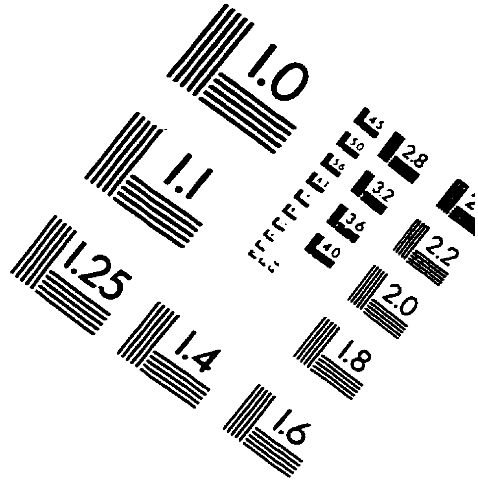
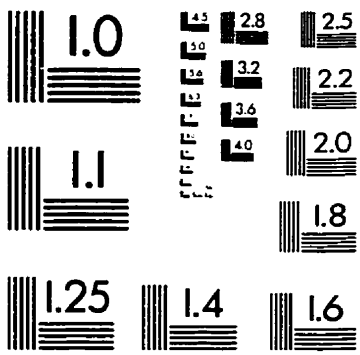
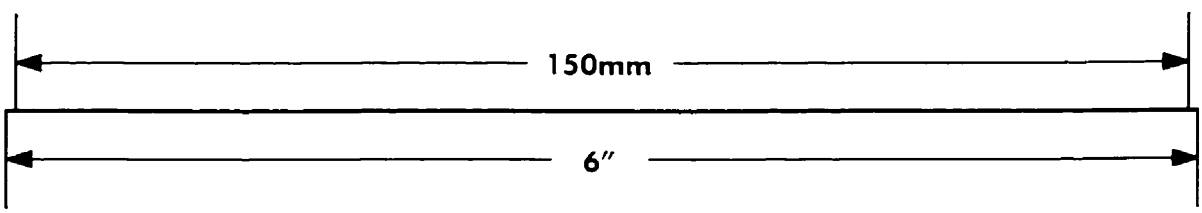
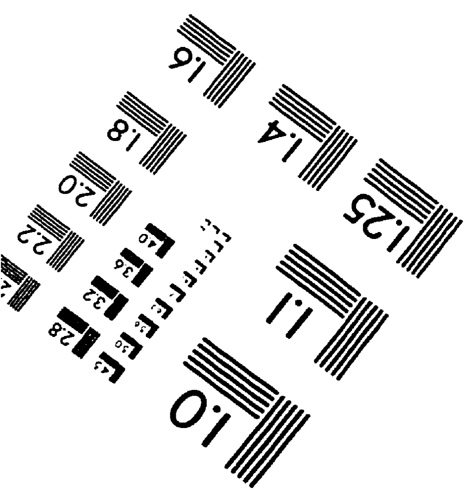
IMAGE EVALUATION TEST TARGET (QA-3)

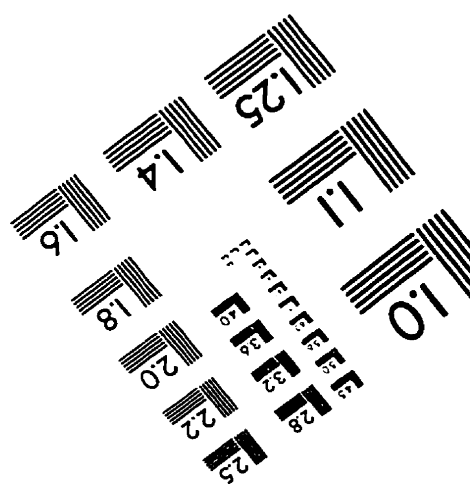












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